CONDITIONING OF PROBLEMS ;

STABILITY OF ALGORITHMS

Assumptions : $f : \mathbb{R} \to \mathbb{R}$

Numbers : $x \, , \, \hat{x}$ (close in value)

$\hat{x} = \text{round}(x)$

$$y = f(x)$$
$$\hat{y} = f(\hat{x})$$

Absolute condition number : $C(x)$

$$|\hat{y} - y| \simeq C(x)\,|x - \hat{x}|$$

Relative condition number : $\mathcal{K}(x)$

$$\left|\frac{\hat{y} - y}{y}\right| \simeq \mathcal{K}(x) \left|\frac{x - \hat{x}}{x}\right|$$

EXAMPLE (Model)

$$\hat{y} - y = f(\hat{x}) - f(x) = \underbrace{\frac{f(\hat{x}) - f(x)}{\hat{x} - x}}_{\simeq f'(x) \text{ as } \hat{x} \to x} (\hat{x} - x)$$

$$C(x) = |f'(x)|$$

$$\frac{\hat{y} - y}{y} = \frac{f(\hat{x}) - f(x)}{f(x)}$$

$$= \underbrace{\frac{f(\hat{x}) - f(x)}{\hat{x} - x}}_{f'(x)} \underbrace{\frac{\hat{x} - x}{x}} \underbrace{\frac{x}{f(x)}}$$

$$\Rightarrow K(x) = \left| \frac{x f'(x)}{f(x)} \right|$$

EXAMPLE

$$f(x) = 2x \quad , \quad f'(x) = 2$$

$$\Rightarrow C(x) = 2 \quad , \quad K(x) = \left| \frac{2 \cdot x}{2x} \right|$$

$$= 1$$

$\Rightarrow$ well-conditioned problem

$$g(x) = \sqrt{x} \quad , \quad g'(x) = \frac{1}{2} \frac{1}{\sqrt{x}}$$

$$\Rightarrow C(x) = \frac{1}{2} \frac{1}{\sqrt{x}} \longrightarrow$$

if x is small
e.g. $x \sim 10^{-8}$
$\Rightarrow C(x) \sim 10^{4}$

$$\Rightarrow K(x) = \frac{1}{2}$$

STABILITY

$$fl(x+y) \equiv \text{round}(x) \oplus \text{round}(y)$$

$$= (x(1+\delta_1) + y(1+\delta_2))(1+\delta_3)$$

Forward error analysis (FEA) :

How far are we from the true solution?

Backward error analysis (BEA) :

Given the answer, what was the problem?

NEWTON'S METHOD

Initial value : x_0

Iteration : $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

Taylor-polynomial connection :

$$f(x) = f(x_0) + (x - x_0) f'(x_0)$$

$$+ \frac{(x - x_0)^2}{2} f''(\xi) \, , \quad \xi \in [x_0, x] .$$

Let x_* be s.t. $f(x_*) = 0$.

Drop the 2^{nd} order term and denote $x_1 = x_*$

Therefore :

$$0 = f(x_0) + (x_1 - x_0) f'(x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

THEOREM If $f \in C^2$ and x_0 is sufficiently good, and $f'(x_*) \neq 0$, then Newton's method converges quadratically. (asymptotic)

<u>Proof</u> It follows from the Taylor expansion that

$$x_* = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{(x_* - x_k)^2}{2} \frac{f''(\xi_k)}{f'(x_k)}$$

Take x_{k+1} from the method and substract

$$x_{k+1} - x_* = \underbrace{\frac{f''(\xi_k)}{2f'(x_k)}}_{|\cdot| \leq D \text{ (const)}} (x_k - x_*)^2$$

$\square$

In other words: $\frac{x_{k+1} - x_*}{(x_k - x_*)^2} = \text{const}$

as $x_k \longrightarrow x_*$.

What happens if $f'(x_*) = 0$?

$$x_{k+1} - x_* = \frac{f''(\xi_k)}{\underbrace{2f'(x_k)}_{\longrightarrow 0, \text{ as } x_k \to x_*}} (x_k - x_*)^2$$

Taylor : $f'(x_k) = \underbrace{f'(x_*)}_{=0} + (x_k - x_*) f''(\eta_k)$

$$= (x_k - x_*) f''(\eta_k)$$

$$\Rightarrow \quad x_{k+1} - x_* = \frac{f''(\xi_k)}{2f''(\eta_k)} (x_k - x_*)$$

The method has degenerated to a linear method !

EXAMPLE $f(x) = x^2$, $f'(x) = 2x$

Newton : $x_{k+1} = x_k - \dfrac{x_k^2}{2x_k} = \dfrac{1}{2} x_k$

FIXED-POINT ITERATION

For instance: Newton

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Let $x_k = x_*$, i.e., $f(x_*) = 0$, then

$$x_* = x_* - \underbrace{\frac{0}{f'(x_*)}}_{=0} = \varphi(x_*)$$

There are many forms: $f(x) = 0$

$$\varphi(x) \equiv x - f(x) = x$$
$$\varphi(x) \equiv x + f(x) = x$$

THEOREM Let x_* be the fixed-point. Assume that $\varphi \in C^1$ je $|\varphi'(x)| < 1$ on $[x_* - \delta, x_* + \delta]$. If x_0 is on the interval then the iteration converges.