Proof Fixed point x_*.

$$x_{k+1} = \varphi(x_k) \qquad \text{Taylor :}$$

$$= \varphi(x_*) + (x_k - x_*)\varphi'(\xi_k)$$

$$= x_* + (x_k - x_*)\varphi'(\xi_k)$$

But,

$$\underbrace{x_{k+1} - x_*}_{e_{k+1}} = \underbrace{(x_k - x_*)}_{e_k}\varphi'(\xi_k)$$

$$\Rightarrow |e_{k+1}| < \underbrace{|\varphi'(\xi_k)|}_{<1}|e_k|$$

$\square$

How to estimate the rate of convergence from observed data?

General setting : Data y_i; Point x_i

Rate α

Model: $y_i = C x_i^{\alpha}$

$$\log y = \log C x^{\alpha} = \log C + \log x^{\alpha} = \log C + \underline{\underline{\alpha}} \log x$$

INTERPOLATION

IDEA: Approximate a function $f(x)$ over $x \in [a, b]$ with a polynomial $p(x)$ such that in data points (x_i, y_i), $i = 0, 1, \dots, n$, the approximation is exact:

$$y_i = p(x_i), \quad \text{for all } i.$$

EXAMPLE $(1,2), (2,3), (3,6)$ $\{(x_i, y_i),\ i = 0, 1, 2\}$

Interval: $[1, 3]$; $p_2(x) = \sum_{j=0}^{2} c_j x^j$

2^{nd} order polynomial $\Rightarrow$ 3 unknown coefficients

Matrix formulation: (Vandermonde)

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \qquad \text{i.e.}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

$\Rightarrow c_0 = 3, \ c_1 = -2, \ c_2 = 1;$

$$p_2(x) = x^2 - 2x + 3$$

Computational complexity of linear solution: $\Theta(n^3)$

We now have the polynomial in the natural basis.

Question: What would be the ideal basis?

$$p(x) = \sum y_i \varphi_i(x), \quad \text{where} \begin{cases} \varphi_i(x_i) = 1 \\ \varphi_i(x_j) = 0 \\ i \neq j \end{cases}$$

DEFINITION

LAGRANGE BASIS POLYNOMIALS ; $x_i \neq x_j$, $i \neq j$

$$\varphi_i(x) = \prod_{i \neq j} \frac{(x - x_j)}{x_i - x_j}$$

LAGRANGE INTERPOLATING POLYNOMIAL

$$p(x) = \sum_{i=0}^{n} y_i \varphi_i(x)$$

EXAMPLE $(1,2)$, $(2,3)$, $(3,6)$

$$\varphi_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} ; \quad \varphi_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3}$$

$$\varphi_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} ; \quad P_2(x) = 2 \cdot \varphi_0(x) + 3\varphi_1(x) + 6\varphi_2(x)$$

$$= x^2 - 2x + 3$$

$\Theta(n^2)$

NEWTON'S INTERPOLATION

IDEA: Extend the natural basis:

$$1, \quad x - x_0, \quad (x - x_0)(x - x_1), \quad \ldots, \quad \prod_{j=0}^{n-1} (x - x_j)$$

DEFINITION NEWTON'S INTERPOLATING POLYNOMIAL

$$p_n(x) = a_0 + a_1 (x - x_0) + \ldots + a_n \prod_{j=0}^{n-1} (x - x_j)$$

The coefficients are chosen so that $p_n(x_i) = y_i$.

$$p(x_0) = y_0 \implies a_0 = y_0$$

$$p(x_1) = a_0 + a_1 (x_1 - x_0) = y_1$$

$$\implies a_1 = \frac{y_1 - a_0}{x_1 - x_0}$$

Linear system (lower triangular)

$$\begin{pmatrix}
1 & 0 & \ldots & & \\
1 & x_1 - x_0 & 0 & \ldots & \\
1 & x_2 - x_0 & (x_2 - x_0)(x_2 - 1) & 0 & \ldots \\
\vdots & & & & 0 \\
1 & x_n - x_0 & \ldots & & \displaystyle\prod_{j=0}^{n-1} (x_n - x_j)
\end{pmatrix}$$

There may be floating point issues:
potential underflow or overflow.

EXAMPLE $\quad P_2(x) = a_0 + a_1(x-1) + a_2(x-1)(x-2)$

System: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

$\Rightarrow a_0 = 2, \quad a_1 = 1, \quad a_2 = 1$

$\Rightarrow P_2(x) = x^2 - 2x + 3$

THEOREM Interpolating polynomial is unique.

Proof (IDEA)

$P_n(x) \longrightarrow n$ roots

Let $p_n(x)$ and $q_n(x)$ be two interpolating polynomials.

$$P_n(x_j) - q_n(x_j) = 0, \quad j = \underbrace{0, 1, \dots, n}_{n+1 \text{ zeros}}$$

$\Rightarrow P_n(x) = q_n(x)$

$\square$