NEWTON'S INTERPOLATION

IDEA: Extend the natural basis:

$$1, \quad x - x_0, \quad (x - x_0)(x - x_1), \quad \dots, \quad \prod_{j=0}^{n-1} (x - x_j)$$

DEFINITION NEWTON'S INTERPOLATING POLYNOMIAL

$$p_n(x) = a_0 + a_1 (x - x_0) + \dots + a_n \prod_{j=0}^{n-1} (x - x_j)$$

The coefficients are chosen so that $p_n(x_i) = y_i$.

$$p(x_0) = y_0 \quad \Rightarrow \quad a_0 = y_0$$

$$p(x_1) = a_0 + a_1 (x_1 - x_0) = y_1$$

$$\Rightarrow a_1 = \frac{y_1 - a_0}{x_1 - x_0}$$

Linear system (lower triangular)

$$\begin{pmatrix} 1 & 0 & \dots & & \\ 1 & x_1 - x_0 & 0 & \dots & \\ 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - 1) & 0 & \dots \\ \vdots & & & & 0 \\ 1 & x_n - x_0 & \dots & & \underline{\underline{\prod_{j=0}^{n-1} (x_n - x_j)}} \end{pmatrix}$$

DIVIDED DIFFERENCES

Newton: $1, \; x - x_0, \; (x-x_0)(x-x_1), \; \ldots, \; \ldots$

$$p(x) = a_0 + a_1(x-x_0) + a_2(\quad)(\quad) + \ldots$$

$$\ldots + a_n \prod_{j=0}^{n-1} (x - x_j)$$

Natural basis:

$$p(x) = \sum_{k=0}^{n} c_k x^k \; ; \quad \text{Connection: } c_n = a_n$$

<u>Definition</u> Divided difference of order k

$$f[x_0, x_1, \ldots, x_k] = a_k$$

THEOREM

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

Notice: The recursion terminates, because

$$f[x_i] = y_i = f_i$$

EXAMPLE $(1,2),\ (2,3),\ (3,6)$

$$P_2(x) = x^2 - 2x + 3$$

Newton : $a_0 = \underline{2}$, $a_1 = \underline{\underline{1}}$, $a_2 = \underline{\underline{\underline{1}}}$

$f[x_0] = \underline{2}$

$f[x_1] = 3$ $\quad f[x_0, x_1] = \dfrac{3-2}{2-1} = \underline{\underline{1}}$

$f[x_2] = 6$ $\quad f[x_1, x_2] = \dfrac{6-3}{3-2} = 3$ $\quad f[x_0, x_1, x_2] = \dfrac{3-1}{3-1}$

$= \underline{\underline{\underline{1}}}$

We get the Newton coefficients !

Why does this work ?

One point : $f[x_j] = f_j = y_j$

Two points : $f[x_i, x_j] = \underbrace{\dfrac{f[x_j] - f[x_i]}{x_j - x_i}}$

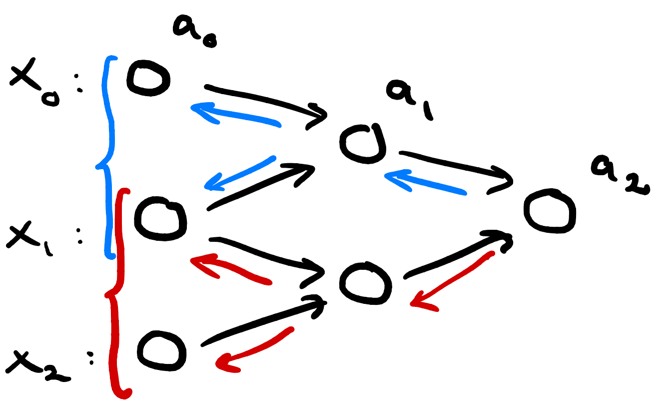
Line spanning two points :

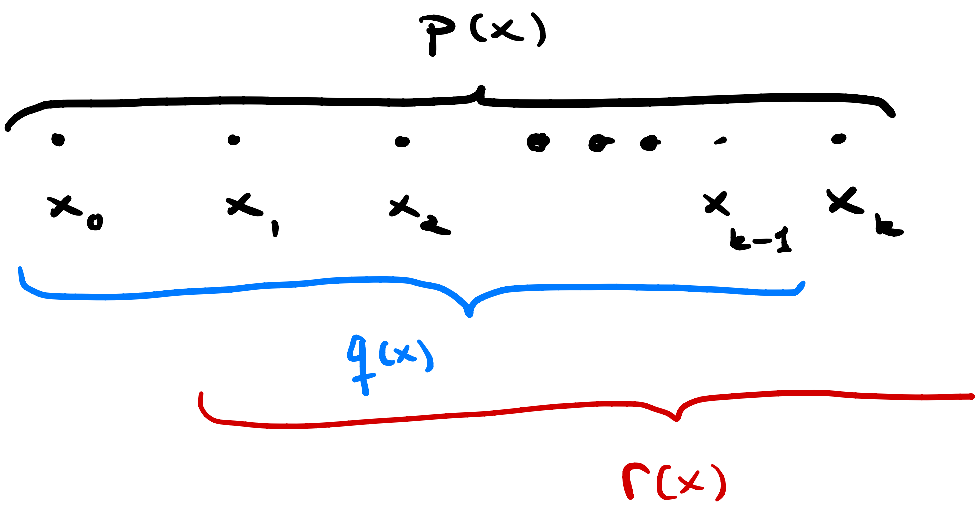
$$y - y_0 = \underbrace{\frac{y_1 - y_0}{x_1 - x_0}}(x - x_0)$$

Proof

three interpolating polynomials:

$p(x)$	degree	k
$q(x)$	"	$k-1$
$r(x)$	"	$k-1$





Claim: $$p(x) = q(x) + \frac{x - x_0}{x_k - x_0} \underbrace{\left(r(x) - q(x) \right)}_{=0 \text{ for } x_i}$$

x_0 : $p(x_0) = q(x_0) = f_0$

$x_1, \dots, x_{k-1}$: $p(x_i) = q(x_i)$

x_k : $$p(x_k) = q(x_k) + \underbrace{\frac{x_k - x_0}{x_k - x_0}}_{=1} \left(r(x_k) - q(x_k) \right)$$

$$= r(x_k)$$

The highest order term has coeff : $\dfrac{r(x) - q(x)}{x_k - x_0}$ $\square$

INTERPOLATION ERROR : $R(x) = f(x) - p(x)$

Assumptions : $f \in C^{n+1}$ ($n+1$ derivatives continuous)

Data : (x_i, y_i) ; $x' \neq x_i$

Auxiliary function :

$$h(x) = f(x) - p(x) - c\,w(x)$$

$$w(x) = \prod_{j=0}^{n} (x - x_j)$$

$$c = \frac{f(x') - p(x')}{w(x')}$$

Verify zeros : $x_i \longrightarrow h(x_i) = 0$

$$x' : h(x') = f(x') - p(x') - \frac{(f(x') - p(x'))}{\cancel{w(x')}}\cancel{w(x')}$$

$$= 0$$

There are at least $n+2$ zeros.

Rolle's Theorem : $h^{(n+1)}$ will have at least one zero

$\longrightarrow$ Let's call this point ξ

$$h^{(n+1)}(x) = f^{(n+1)}(x) - \underbrace{p^{(n+1)}(x)}_{=0} - c\,w^{(n+1)}(x)$$

$$= f^{(n+1)}(x) - c\,(n+1)!$$

$$\Rightarrow \quad h^{(n+1)}(\xi) = f^{(n+1)}(\xi) - c\,(n+1)! = 0$$

$$\Rightarrow \quad c = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

Residual at x':

$$R(x') = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x' - x_j)$$

THEOREM

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j),$$

where $\xi = \xi(x)$.

Notice: Constant c is a divided difference:

$$f[x_0, x_1, \ldots, x_n, x] = \frac{1}{(n+1)!} f^{(n+1)}(\xi(x))$$

Error amplification factor : $K(x)$

Measurements : "True value" f_i

"Wrong value" $\hat{f}_i$

$$p(x) - \hat{p}(x) = \sum_{i=0}^{n} \underbrace{(f_i - \hat{f}_i)}_{} L_i(x)$$

Assuming : $|f_i - \hat{f}_i| < \delta$, $i = 0, \ldots, n$

then

$$|p(x) - \hat{p}(x)| \leq \delta \underbrace{\sum_{i=0}^{n} |L_i(x)|}_{}$$

$$= \delta K(x)$$

$K(x)$ can be used to estimate the accumulated error in interpolation.

Evaluation of polynomials :

Horner's Rule: Complexity is $\Theta(n)$.

$$y = c_n \;;\quad y = yx + c_{n+1}$$

$$\Rightarrow \quad y = \sum_{j=0}^{n} c_j x^j$$