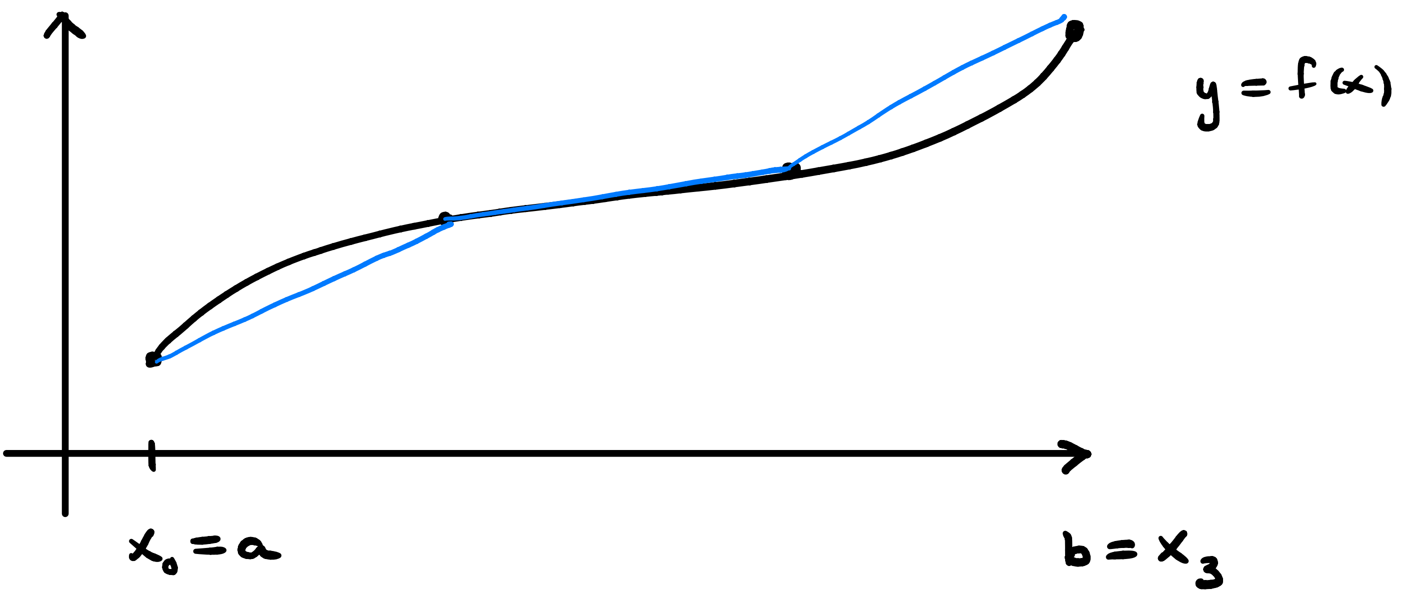
PIECEWISE INTERPOLATION



Setup: Interval $[a, b]$

$h = (b-a)/n$, where n is the number of subintervals

IDEA:

Approximate the function on each subinterval using some <u>low order</u> interpolating polynomial.

Linear case:

$$l_i(x) = f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

$x \in [x_{i-1}, x_i]$

Interpolation error :

$$f(x) - l_i(x) = \frac{f''(\xi)}{2!}(x - x_{i-1})(x - x_i)$$

Assumption : $|f''(x)| \leq M$ (const.) :

$$|f(x) - l_i(x)| \leq M\frac{h^2}{8}, \quad x \in [x_{i-1}, x_i]$$

$\hookrightarrow$ through maximization

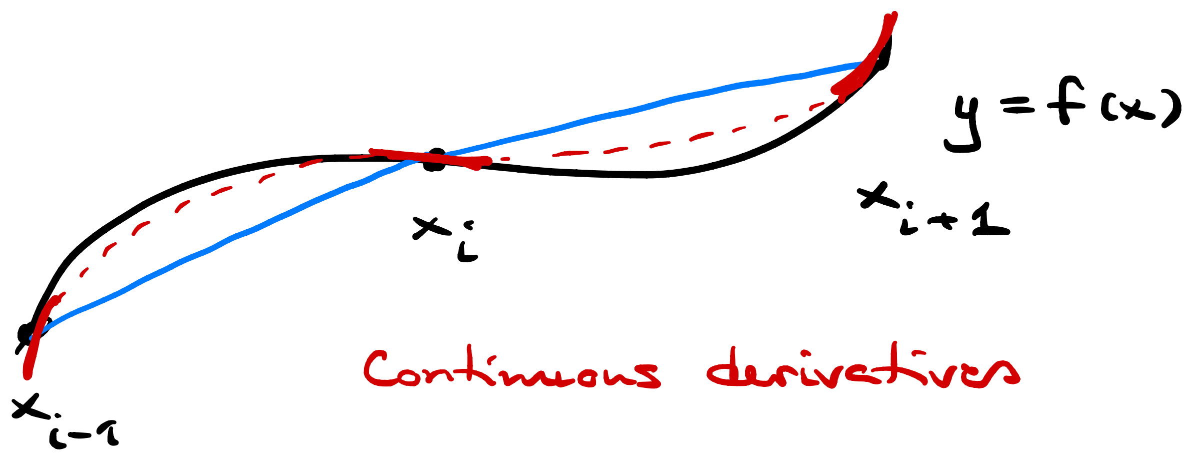
Notice :

If f'' is bounded over the whole interval $[a, b]$ then the error is the same over the whole interval.

Hermite interpolation : degree 3

We need four constraints : $P_3(x) = \sum_{j=0}^{3} c_j x^j$

Piecewise Linear :



Polynomial: $p(x)$

$p'(x)$ is a quadratic polynomial

$$p'(x) = f'(x_{i-1}) \frac{x - x_i}{\underline{x_{i-1} - x_i}} + f'(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

$$+ \alpha (x - x_{i-1})(x - x_i)$$

We have: $p'(x_i) = f'(x_i)$; $p'(x_{i-1}) = f'(x_{i-1})$

Integrating: $h = x_i - x_{i-1}$

$$p(x) = - \frac{f'(x_{i-1})}{\underline{h}} \int_{x_{i-1}}^{x} (t - x_i)\, dt$$

$$+ \frac{f'(x_i)}{h} \int_{x_{i-1}}^{x} (t - x_{i-1})\, dt$$

$$+ \alpha \int_{x_{i-1}}^{x} (t - x_{i-1})(t - x_i)\, dt + C$$

Conditions: $p(x_{i-1}) = f(x_{i-1}) \Rightarrow C = f(x_{i-1})$

$p(x_i) = f(x_i)$

$$\Rightarrow \quad \alpha = \frac{3}{h^2}\left(f'(x_{i-1}) + f'(x_i)\right)$$

$$+ \frac{6}{h^3}\left(f(x_{i-1}) - f(x_i)\right)$$

SPLINES : $S(x)$

(1) We do not impose continuity for derivatives.

(2) We get piecewise polynomial construction with continuous 2^{nd} derivatives; cubic

This requires a global step :

all coefficient are defined first ; only evaluation is piecewise

Setup : $h = x_i - x_{i-1}$ $\quad (= \text{vakio})$

$$z_i = S''(x_i), \quad i = 1, \ldots, n-1$$

Interval : $[x_{i-1}, x_i] \longrightarrow$ interval i

Now : $$S''(x_i) = \frac{1}{h} z_{i-1}(x_i - x) + \frac{1}{h} z_i (x - x_{i-1})$$

Integrating (twice):

$$s_i(x) = \frac{1}{h} z_{i-1} \frac{(x_i - x)^3}{6} + \frac{1}{h} z_i \frac{(x - x_{i-1})^3}{6}$$

$$+ C_i (x - x_{i-1}) + D_i$$

We get: $D_i = f_{i-1} - \frac{h^2}{6} z_{i-1}$

$$C_i = \frac{1}{h}\left[f_i - f_{i-1} + \frac{h^2}{6}(z_{i-1} - z_i) \right]$$

Notice: $f(x_i) = f_i$

Also: $s(x)$ has now been defined over the subintervals!

But(!) z_i:s are still unknown!

Let us compute the derivatives of s and use continuity: $s_i'(x_i) = s_{i+1}'(x_i)$

$$\frac{h}{2} z_i + \frac{1}{h}(f_i - f_{i-1}) + \frac{h^2}{6}(z_{i-1} - z_i) =$$

$$- \frac{h}{2} z_i + \frac{1}{h}(f_{i+1} - f_i) + \frac{h^2}{6}(z_i - z_{i+1}),$$

$$i = 1, \ldots, n-1$$

This is in fact a tridiagonal system:

$$\frac{2h}{3} z_i + \frac{h}{6} z_{i-1} + \frac{h}{6} z_i =$$

$$\frac{1}{h}\left(f_{i+1} - 2f_i + f_{i+1} \right) = b_i$$

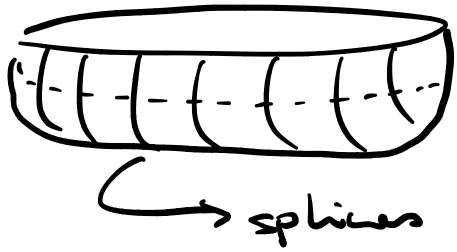
z_0 & z_n have to be moved to the RHS:

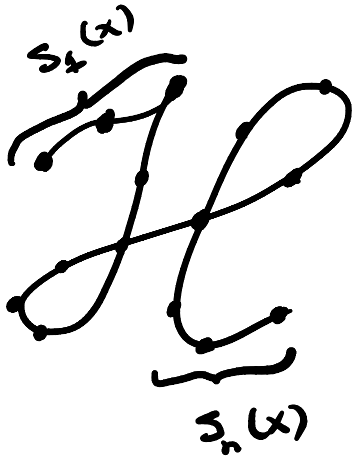
$$b_1 = \frac{1}{h}\left(f_2 - 2f_1 + f_0 \right) - \frac{h}{6} z_0$$

$$b_{n+1} = \frac{1}{h}\left(f_n - 2f_{n-1} + f_{n-2} \right) - \frac{h}{6} z_n$$

The so-called natural spline: $z_0 = z_n = 0$

"Model Boat"





Application: Font design (numerically)