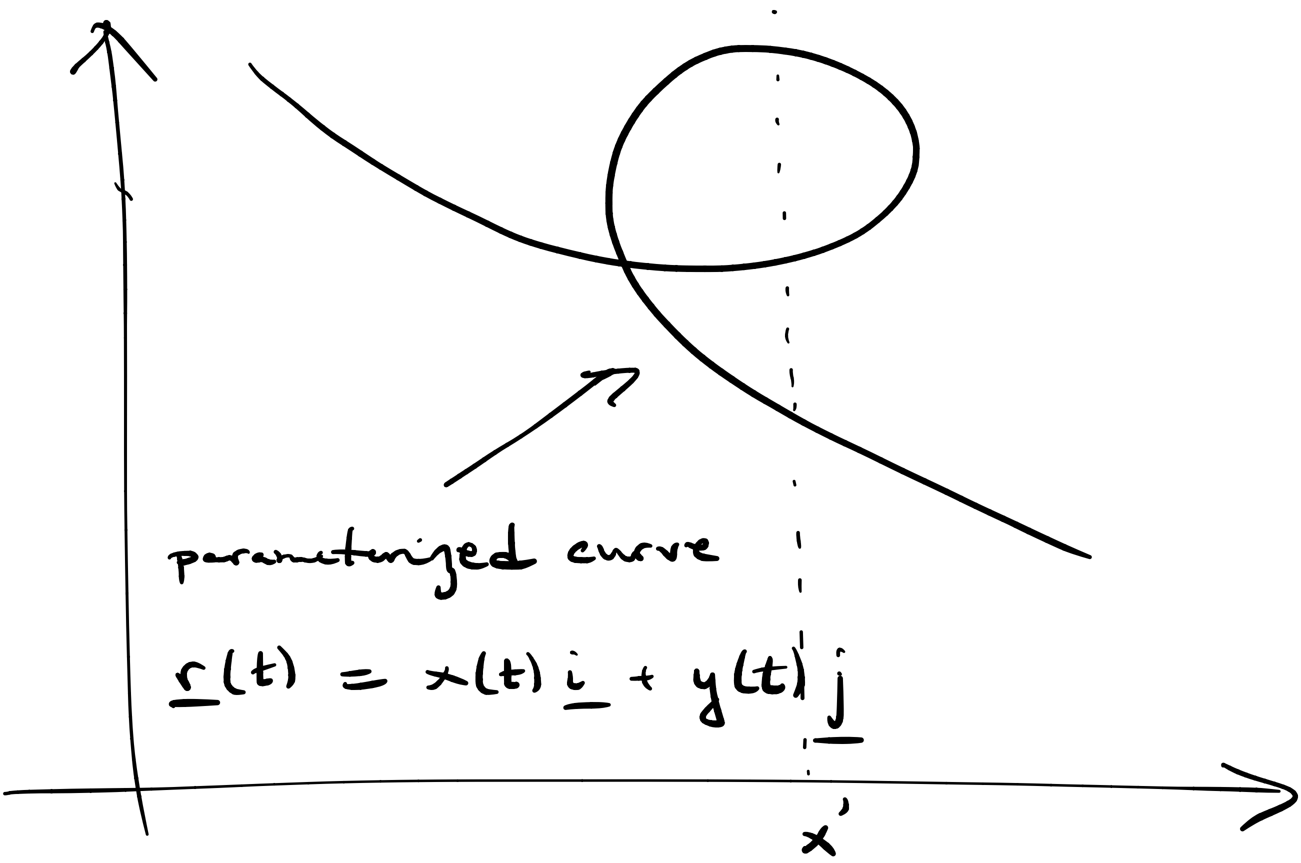
BEZIER CURVES : Parameterized curves



Bernstein Polynomials : $B_k^n(t)$, $t \in [0,1]$

DEFINITION $\quad B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k}$

Properties:

1) $\sum_{k=0}^{n} B_k^n(t) = 1 \qquad \left(= (t + 1 - t)^n \right)$

2) $0 \leq B_k^n(t) \leq 1$, $\forall\, k, n \geq 0$

3) $B_0^n(0) = B_n^n(1) = 1$, otherwise $B_k^n(0) = B_k^n(1) = 0$

Combinatorics :

$$B_k^n(t) = (1-t)\,B_k^{n-1}(t) + t\,B_{k-1}^{n-1}(t)$$

<u>Bezier Curves</u> : Control points $x_k \in \mathbb{R}^n$

DEFINITION Convex Hull

Let $x = \{x_1, \dots, x_k\}$ be a set of points in $\mathbb{R}^n$.

The convex hull is the set

$$C_H(x) = \left\{ y \in \mathbb{R}^n \;\middle|\; y = \sum_{i=1}^{k} a_i x_i,\ a_i \geq 0,\ \sum_{i=1}^{k} a_i = 1 \right\}$$

DEFINITION BEZIER CURVE

$$\beta^n(t) = \sum_{k=0}^{n} x_k B_k^n(t)$$

Sanity check : $t=0$, $B_k^n(0) = 0$, except $B_0^n(0) = 1$

$$\Rightarrow \beta^n(0) = x_0$$

$$\Rightarrow \beta^n(1) = x_n$$

<u>Closed curves</u> : $x_0 = x_n$

What about the continuous tangents?

$$\frac{d}{dt} B_k^n(t) = \binom{n}{k}\left(\underline{k}\, t^{k-1} (1-t)^{n-k} - \underline{(n-k)}\, t^k (1-t)^{n-k-1} \right)$$

$$\left[\binom{n}{k} = \frac{n!}{\underline{k}!\,\underline{(n-k)}!} \right]$$

$$= n \left[\frac{(n-1)!}{(k-1)!\,(n-k)!} t^{k-1} (1-t)^{n-k} - \frac{(n-1)!}{k!\,(n-k-1)!} t^k (1-t)^{n-k-1} \right]$$

$$= n \left(B_{k-1}^{n-1}(t) - B_k^{n-1}(t) \right)$$

Therefore

$$\frac{d}{dt} \beta^n(t) = n \sum_{k=0}^{n} \left(B_{k-1}^{n-1}(t) - B_k^{n-1}(t) \right) x_k$$

$$= n \left[\sum_{k=1}^{n} B_{k-1}^{n-1}(t)\, x_k - \sum_{k=0}^{n-1} B_k^{n-1}(t)\, x_k \right]$$

$$= n \left(\sum_{k=0}^{n-1} B_k^{n-1}(t)\, x_{k+1} - \sum_{k=0}^{n-1} B_k^{n-1}(t)\, x_k \right)$$

$$= \underbrace{n \sum_{k=0}^{n-1} (x_{k+1} - x_k)\, B_k^{n-1}(t)}_{\text{Bezier}}$$

Hence, for the closed curves:

$$\begin{cases} \frac{d}{dt} \beta^n(0) = n(x_1 - x_0) \\ \frac{d}{dt} \beta^n(1) = n(x_n - x_{n-1}) \end{cases}$$

$\Rightarrow$ for smoothness $x_1 - x_0 \parallel x_n - x_{n-1}$

<u>Lifting</u> : Control points define the curve but the converse is not true.

Consider:

$$\beta^n(t) = \sum_{k=0}^{n} B_k^n(t)\, x_k$$

$$= \sum_{k=0}^{n+1} B_k^{n+1}(t)\, y_k = \alpha^{n+1}(t)$$

Let us use the convention $x_{-1} = x_{n+1} = 0$.
We get the condition:

$$y_k = \left(1 - \frac{k}{n+1}\right) x_k + \frac{k}{n+1} x_{k-1} .$$

De Casteljau Algorithm

Control points: $x_0, x_1, \ldots, x_n$.

(1) Define constant curves: $\beta_i^0(t) = x_i$

(2) $\beta_i^r(t) = (1-t)\beta_i^{r-1}(t) + t\beta_{i+1}^{r-1}(t)$,

$$r = 1, \ldots, n, \quad i = 0, \ldots, n-r$$

The algorithm terminates at $\beta_0^n(t)$.