NUMERICAL INTEGRATION

Integration schemes are called quadratures. Therefore, numerical integration methods are simply numerical quadratures.

Notice: There are no simple integration schemes in higher dimensions; already 2D-cases are complicated.

MONTE CARLO : Randomised quadrature rules

Central Limit Theorem

Let X_i be i.i.d. random variables; mean μ, variance σ^2.

Then for the mean

$$A_N = \frac{1}{N} \sum_{i=1}^{N} X_i$$

it holds that
$$\operatorname{var}(A_N) = \frac{1}{N^2} \sum_{i=1}^{N} \operatorname{var}(X_i) = \frac{\sigma^2}{N}.$$

In order to get the right unit, we want to consider the standard deviation:

$$\sigma(A_N) = \frac{\sigma}{\sqrt{N}}$$

Consequence:

If our integration problem can be cast into averageing problem, the convergence rate will be

$$\Theta\left(1/\sqrt{N} \right).$$

Notice: The rate is independent of the spatial dimension.

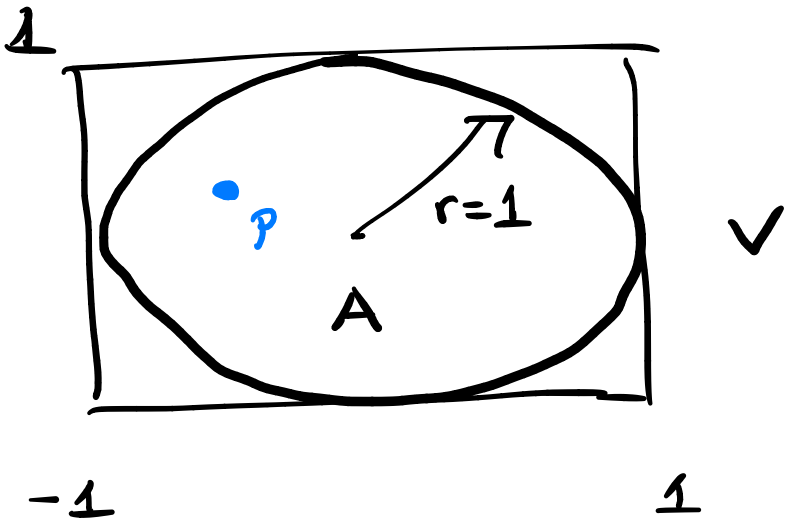
EXAMPLE Estimating the value of π

Area of a circle is $A = \pi r^2$

set $r = 1$

$V = [-1,1] \times [-1,1]$; $|V| = 4$

Ratio of the areas is $\pi/4$.

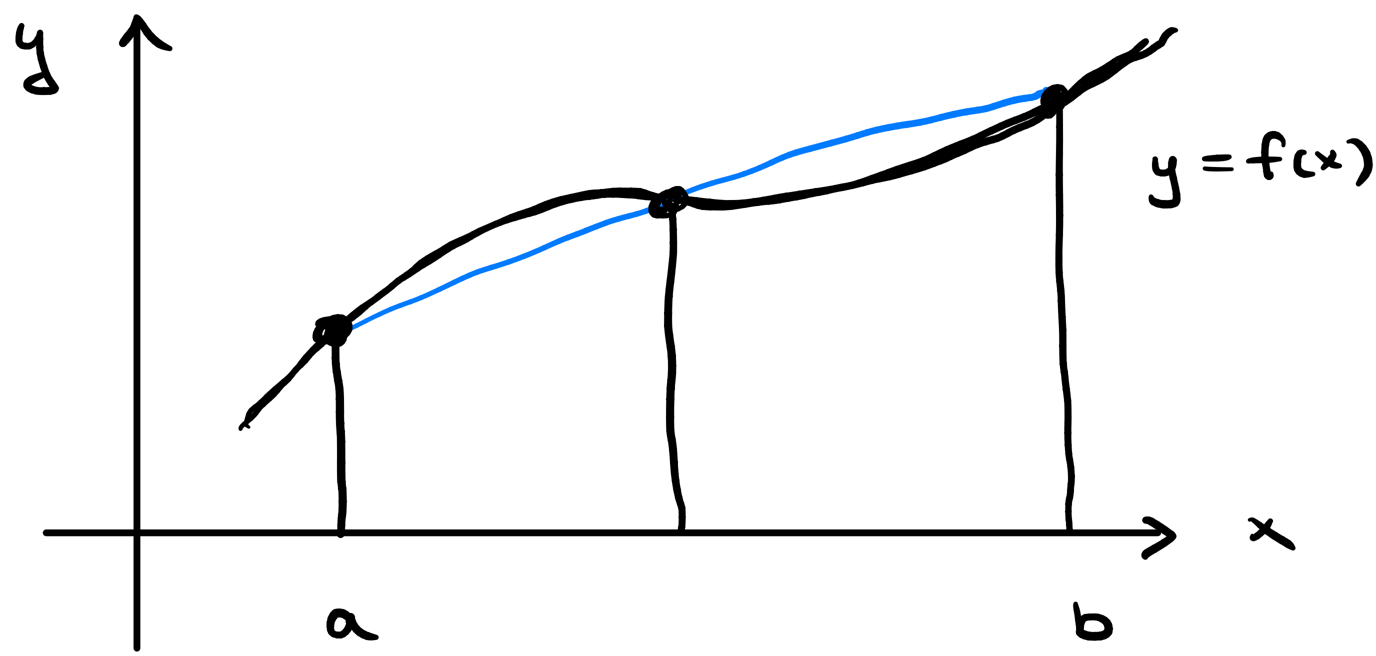


Let $g_i = \begin{cases} 1, & \text{if } p \text{ is inside } A \\ 0, & \text{otherwise} \end{cases}$

Idea: Let us sample points P_i uniformly from V.

In the limit the number of "hits" over all samples tends to the ratio of the areas!

NEWTON-COTES QUADRATURE RULES



Idea: Approximate $\int_a^b f(x)\,dx = I$

$$I \simeq \int_a^b p_k(x)\,dx = Q(p)$$

Where p is an interpolant of f over $[a,b]$

Lagrange:

$$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{n} f(x_i) \int_a^b \left(\prod_{\substack{j=0 \\ i\neq j}}^{n} \frac{x - x_j}{x_i - x_j} \right) dx$$

Let $n = 1$:

$$P_1(x) = f(a)\,\frac{x-b}{a-b} + f(b)\,\frac{x-a}{b-a}$$

so

$$\int_a^b f(x)\,dx \simeq \int_a^b P_1(x)\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right]$$

$\Longrightarrow$ Trapezoidal Rule!

Error formulation:

$$\int_a^b f(x)\,dx - \int_a^b P_1(x)\,dx = \frac{1}{2}\int_a^b f''(\xi)\,(x-a)\cdot(x-b)\,dx$$

Now, $(x-a)(x-b) < 0$ for $x \in (a,b)$

Therefore: $= \dfrac{1}{2} f''(\eta) \displaystyle\int_a^b (x-a)(x-b)\,dx$

Another intermediate value theorem!

$$= -\frac{1}{12}(b-a)^3 f''(\eta)$$

Composite Rule: $h = \dfrac{b-a}{n}$; $x_i = a + ih$,
$i = 0, \dots, n$

$$\int_a^b f(x)\,dx \simeq \frac{h}{2}\Big[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\Big]$$

The total error: $\Theta(h^2) \sim \Theta(1/n^2)$

We say that the method is quadratic.

Let $n=2$: Exact for degree 2 (or lower)

$$\int_a^b f(x)\,dx = A_1 f(a) + A_2 f\left(\frac{a+b}{2}\right) + A_3 f(b)$$

$$\int_a^b 1\,dx = b-a \Rightarrow A_1 + A_2 + A_3 = b-a$$

$$\int_a^b x\,dx = \frac{b^2-a^2}{2} \Rightarrow A_1 a + A_2\left(\frac{a+b}{2}\right) + A_3 b = \frac{b^2-a^2}{2}$$

$$\int_a^b x^2\,dx = \frac{1}{3}(b^3-a^3) \Rightarrow A_1 a^2 + A_2\left(\frac{a+b}{2}\right)^2 + A_2 b^2 = \frac{1}{3}(b^3-a^3)$$

$$A_1 = A_3 = \frac{b-a}{6}$$

$$A_2 = \frac{4(b-a)}{6}$$

This is the so-called Simpson's Rule :

$$\int_a^b f(x)\,dx \simeq \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Composite Rule :

$$\int_a^b f(x)\,dx \simeq \frac{h}{6}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-1}) + f(x_n) \Big]$$

Error : $n=2$: $\frac{1}{2880}(b-a)^5 f^{(4)}(\eta)$

For the composite : $\mathcal{O}(h^4)$

$\Rightarrow$ Exact also for cubic polynomials !