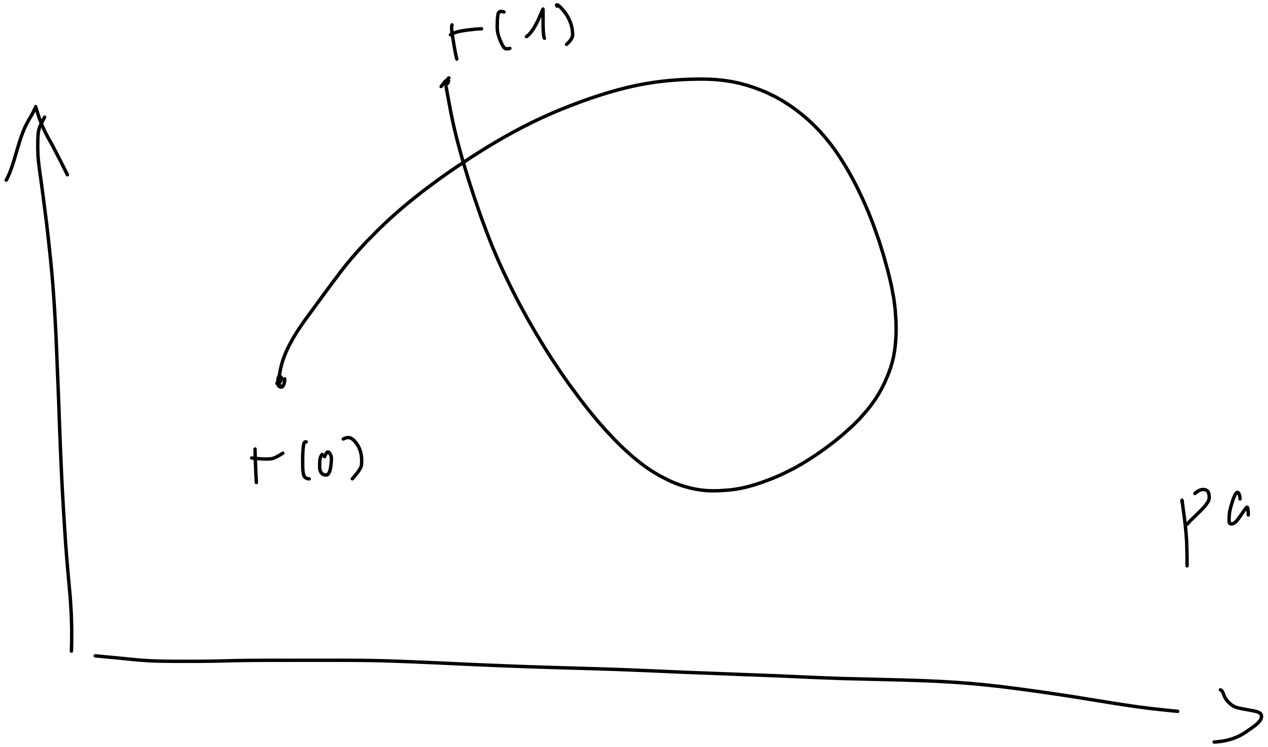
Numerical Analysis, Lecture 7

$\mathbb{R}^2$

Bézier curves

$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



parametrized curve

$r : [0,1] \longmapsto \mathbb{R}^2$, $\vec{r}(t) := x(t)\,\vec{i} + y(t)\,\vec{j}$

$x, y : [0,1] \to \mathbb{R}$

Bernsteins polynomials

Binomial coefficient "n choose k"

Recall $\binom{n}{k} := \frac{n!}{k!\,(n-k)!}$

Binomial formula: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$\binom{n}{0} = 1$, $\binom{n}{n} = 1$, $\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal's triangle

Definition

$t \in [0,1]$

$$B_k^n(t) := \binom{n}{k} t^k (1-t)^{n-k}$$

Properties

Binomial formula

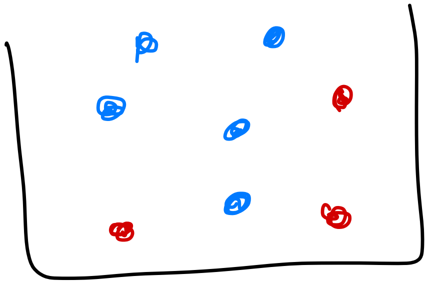
1) $$\sum_{k=0}^{n} B_k^n(t) = (t + 1 - t)^n = 1^n = 1$$

2) $$0 \le B_k^n(t) \le 1$$

$\Rightarrow$ $B_k^n(p)$ is a probability distribution on $\{0, 1, \dots, n\}$ with parameter $p \in [0,1]$

Binomial distribution

k picks with replacement of n picks



$$\frac{\# \bullet}{(\# \bullet + \# \bullet)} = p$$

probability of picking red ball

3) $B_0^n(0) = B_n^n(1) = 1$, otherwise,

$B_k^n(0) = B_k^n(1) = 0$.

Combinatorial formula

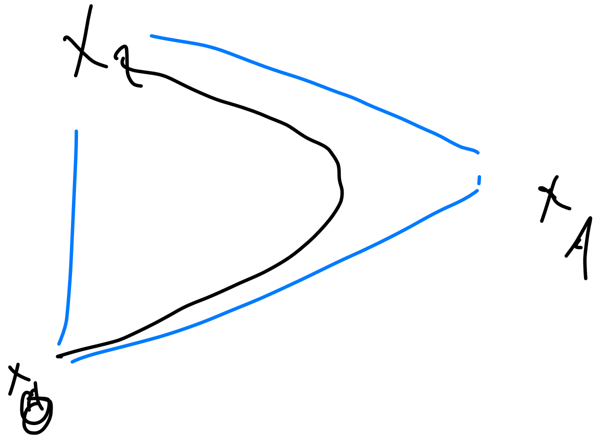
$$B_k^n(t) = (1-t)\,B_k^{n-1}(t) + t\,B_{k-1}^{n-1}(t)$$

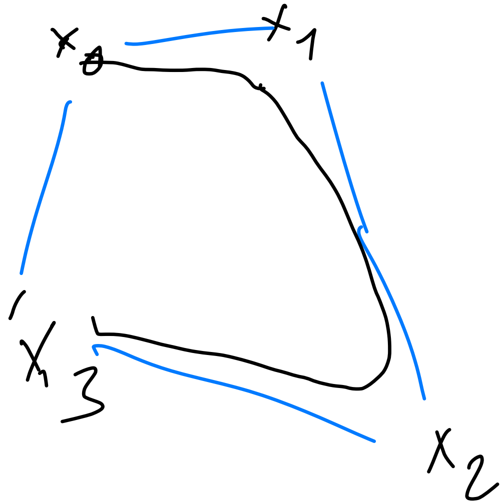
Bézier curves   Control points $x_k \in \mathbb{R}^n$

Definition: Convex hull

Let $X := \{x_0, \dots, x_k\}$ be a finite set of points in $\mathbb{R}^n$.

Simplex (triangle)





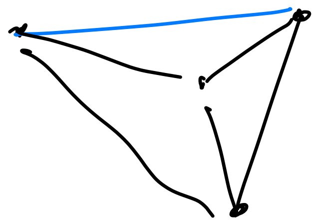
Trapezoid

The convex hull is the set

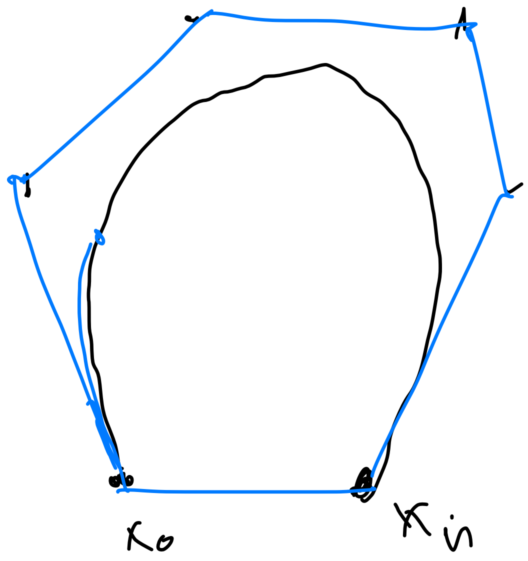
$$\mathrm{Conv}(\mathfrak{X}) := \left\{ y \in \mathbb{R}^n \;\middle|\; y = \sum_{i=0}^{k} \alpha_i x_i ,\; \alpha_i \geq 0,\; \sum_{i=1}^{k} \alpha_i = 1 \right\}$$

Break until

15:00



For a finite sequence of $\underbrace{\text{points } x_0, x_1, \dots, x_n}_{\text{control points}} \in \mathbb{R}^d$



$$B^n : [0,1] \longrightarrow \mathbb{R}^d$$

Sanity check

$B^n(0) = x_0$

$B^n(1) = x_n$

because: For $t = 0$

$B_k^n(0) = 0$, $k \geq 1$,

$B_0^n(0) = 1$.

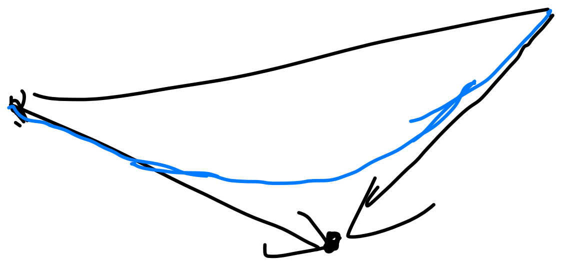
Bézier Curve

$$B^n(t) := \sum_{k=0}^{n} x_k \, B_k^n(t)$$

$\deg B_k^n = n$.

Closed curves $x_0 = x_n$.

Tangents



$$\frac{d}{dt} B_k^n(t)$$

$$= \binom{n}{k} \Big(k\, t^{k-1} (1-t)^{n-k}$$

3 points

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$- (n-k)\, t^k (1-t)^{n-k-1} \Big)$$

$$= n \left[\frac{(n-1)!}{(k-1)!(n-k)!} t^{k-1} (1-t)^{n-k} - \frac{(n-1)!}{k!(n-k-1)!} t^k (1-t)^{n-k-1} \right]$$

$$= n\left[\frac{(n-1)!}{(k-1)!\,(n-k)!}\, t^{k-1}\,(1-t)^{n-k} - \frac{(n-1)!}{k!\,(n-k-1)!}\, t^{k}\,(1-t)^{n-k-1}\right]$$

$$= n\left(B_{k-1}^{n-1}(t) - B_{k}^{n-1}(t)\right)$$

Set: $B_k^m(t) := 0$ if $m < k$.

Recursion formula

$$\frac{d}{dt}\beta^n(t) = n \sum_{k=0}^{n} \left(B_{k-1}^{n-1}(t) - B_{k}^{n-1}(t)\right) x_k$$

Set $B_k^m(t) = 0$ if $k < 0$.

$$= n\left[\sum_{k=1}^{n} B_{k-1}^{n-1}(t)\, x_k - \sum_{k=0}^{n-1} B_{k}^{n-1}(t)\, x_k\right]$$

$$= n\left[\sum_{k=1}^{n} B_{k-1}^{n-1}(t)\,x_k - \sum_{k=0}^{n-1} B_k^{n-1}(t)\,x_k\right]$$

$$= n\left[\sum_{k=0}^{n-1} B_k^{n-1}(t)\,x_{k+1} - \sum_{k=0}^{n-1} B_k^{n-1}(t)\,x_k\right]$$

$$= \underbrace{n\sum_{k=0}^{n-1}\left(x_{k+1}-x_k\right)B_k^{n-1}(t)}_{\text{Bézier}}$$

$$= \frac{d}{dt}\,B^n(t)$$

For closed curves:

$$\frac{d}{dt} \beta^n(0) = n(x_1 - x_0)$$

$$\frac{d}{dt} \beta^n(1) = n(x_n - x_{n-1})$$

For smoothness: need $\frac{d}{dt}\beta^n(0) = \frac{d}{dt}\beta^n(1)$

$$x_1 - x_0 \parallel x_n - x_{n-1}$$

parallel or linearly dependent.

Lifting:

Control points define the curve, but the curve does not define the control points.

$$\beta^n(t) = \sum_{k=0}^{n} B_k^n(t)\, x_k$$

$x_0, \dots, x_n$

$$= \sum_{k=0}^{n+1} B_k^{n+1}(t)\, y_k = \alpha^{n+1}(t)$$

$y_0, \dots, y_{n+1}$

With the convention $x_{-1} = x_{n+1} = 0$,

we get the condition:

$$y_k = \left(1 - \frac{k}{n+1}\right) x_k + \frac{k}{n+1} x_{k-1}$$

y_k is convex combination of x_k and x_{k-1}!

$$\left(1 - \frac{k}{n+1} + \frac{k}{n+1} = 1 \right)$$

Reason for this formula is coming from the <u>De Casteljau algorithm</u>

Control points: $x_0, x_1, \ldots, x_n$

(1) define constant curves

$$\beta_i^0(t) = x_i \quad \text{~~for~~ all } t \in [0,1]$$

(2) $$\beta_i^r(t) = (1-t)\,\beta_i^{r-1}(t) + t\,\beta_{i+1}^{r-1}(t)$$

$$r = 1, \ldots, n \quad , \quad i = 0, \ldots, n-r$$

(Tree recursion!)

one can prove by induction that the algorithm terminates at

$\beta_0^n(t)$, $t \in [0,1]$.

$\binom{n}{2}$ operations

$\Downarrow$

$O\left(\frac{n(n-1)}{2}\right) = O(n^2)$

Algorithm for splitting Bézier curves:

Split $B_0^n(t)$ to Bézier curves

$$B_0^{(0)}, \; B_0^{(1)}, \; \ldots, \; B_0^{(n)}$$

and

$$B_0^{(n)}, \; B_1^{(n-1)}, \; \ldots, \; B_n^{(0)}$$

– Increase the degree

– Split at $t \in [0,1]$

Details: Wikipedia article on Bézier curves.