INITIAL VALUE PROBLEMS (IVP)

Problem:
$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

Assumptions: Existence and uniqueness of the solutions are understood.

In addition: f is continuous and with respect to the solution y is Lipschitz continuous.

For all y_1, y_2, $t \in [a, b]$,

$$|f(t, y_1) - f(t, y_2)| \leq L\,|y_2 - y_1|,$$

where L is a constant, $t_0 \in [a, b]$.

EULER'S METHOD ; step size h (const)

$$y_0 = y(t_0) ;$$

$$y_{k+1} = y_k + h\, f(t_k, y_k),$$

$$k = 0, 1, 2, \ldots$$

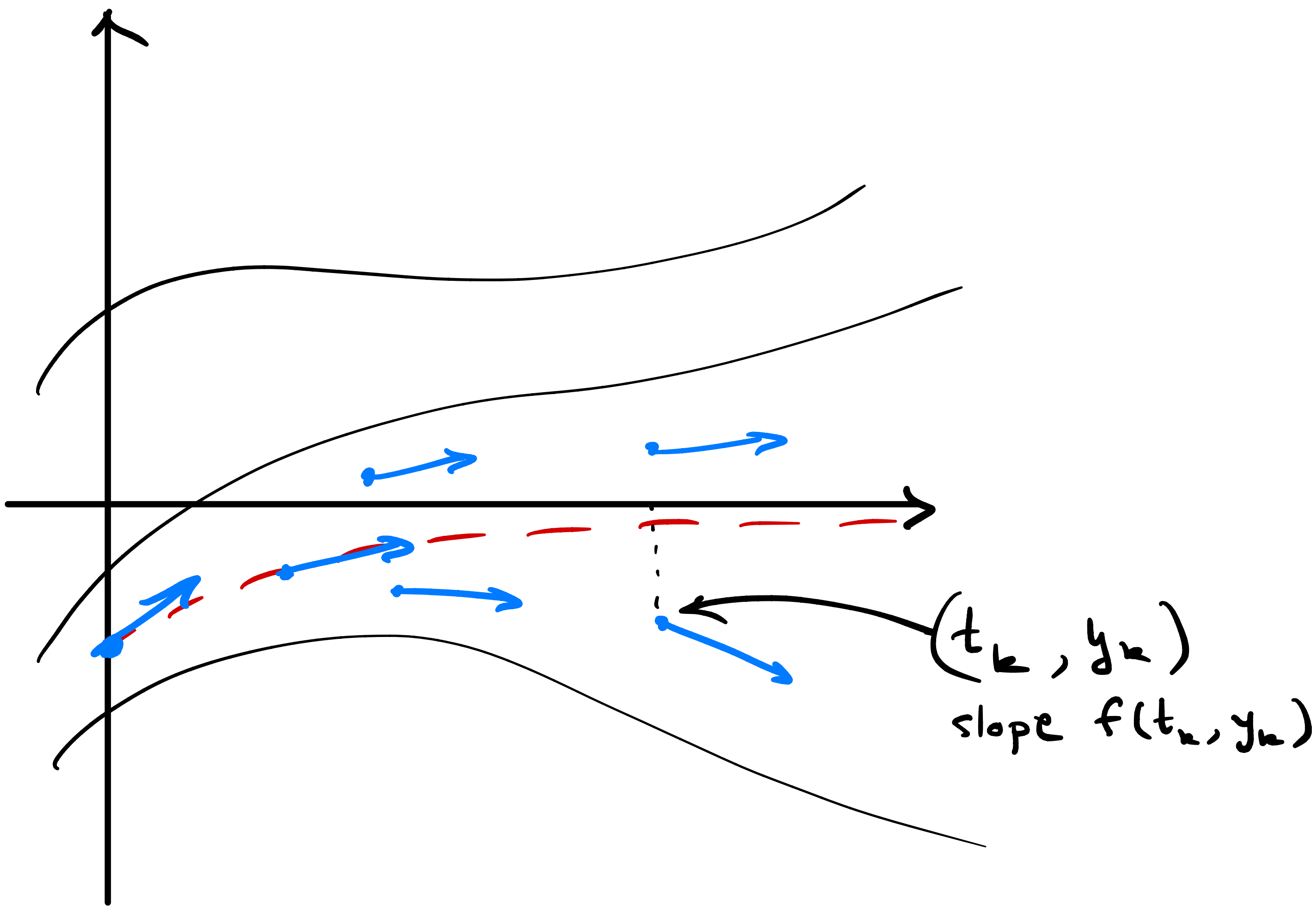
Taylor:

$$y(t_{k+1}) = y(t_k) + h\,y'(t_k) + \frac{h^2}{2} y''(\xi_k)$$

$$= y(t_k) + h\,f(t_k, y(t_k))$$

$$+ \frac{h^2}{2} y''(\xi_k),$$

$$\xi_k \in [a, b]$$



Different types of error:

(A) truncation error (local)

(B) global error

Notation: $y(t_k)$ exact at $t = t_k$

y_k approximate

Euler:
$$\frac{y_{k+1} - y_k}{h} = f(t_k, y_k) + \underbrace{\frac{h^{1}}{2} y''(\xi_k)}_{\substack{\text{local} \\ \text{error} \\ O(h)}}$$

This method is of order 1.

The method is consistent:

$$\lim_{h \to 0} \frac{y_{k+1} - y_k}{h} = y'(t_k) = f(t_k, y(t_k))$$

Global error: At $t = t_k$: $|y(t_k) - y_k| \leq ?$

The method is convergent:

$$\max |y(t_k) - y_k| \xrightarrow{h \to 0} 0$$

THEOREM Euler's method is convergent.

<u>Proof</u> Let $d_j = y(t_j) - y_j$

Taylor – Euler :

$$d_{k+1} = d_k + h\left[f(t, y(t_k)) - f(t_k, y_k)\right] + \frac{h^2}{2} y''(\xi_k)$$

Assume Lipschitz and

$|y''(t_k)| \le M$: We get

$$|d_{k+1}| \le |d_k| + hL|d_k| + \frac{h^2}{2}M$$
$$= (1 + hL)|d_k| + \frac{h^2}{2}M$$

Estimate: $\gamma_{k+1} \le (1+\alpha)\gamma_k + \beta$, $\alpha, \beta > 0$

$$\Rightarrow \gamma_n \le e^{n\alpha}\gamma_0 + \frac{e^{n\alpha} - 1}{\alpha}\beta$$

Why: $\gamma_n \le (1+\alpha)^2 \gamma_{n-2} + [(1+\alpha) + 1]\beta$

$$= (1+\alpha)^n \gamma_0 + \left[\sum_{j=0}^{n-1} (1+\alpha)^j\right]\beta$$

Note: $(1+\alpha) \le e^{\alpha} = 1 + \alpha + \underbrace{\frac{\alpha^2}{2} e^{\xi}}_{>0}$

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Together:

$$|d_{k+1}| \le e^{(k+1)hL} |d_0| + \frac{e^{(k+1)hL} - 1}{L} \frac{h}{2} M$$

Now $T \in [a,b]$, $T > t_0$.

$kh \le T - t_0$:

It is assumed that $y_0 \to y(t_0)$ as $h \to 0$.

$$\max_k |d_k| \le e^{L(T-t_0)} |d_0| + \frac{e^{L(T-t_0)} - 1}{L} \frac{h}{2} M$$

$h \to 0 \implies$ method is convergent.

Global error is $O(h)$.

$\square$

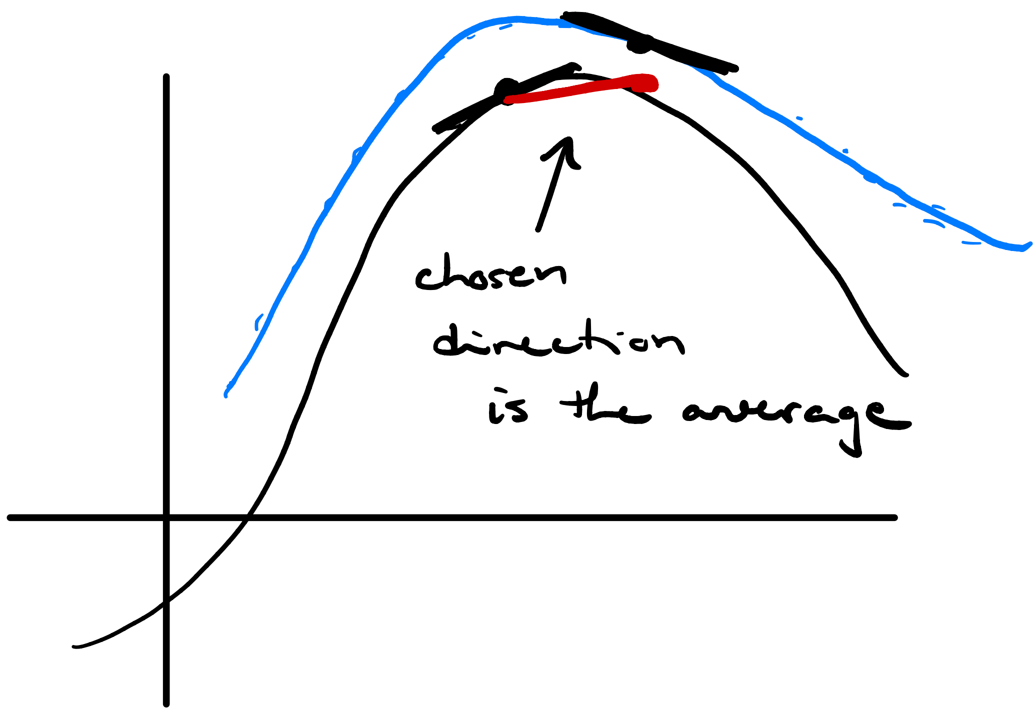
Idea: Predictor - corrector

__Heun's Method__

$$\tilde{y}_{k+1} = y_k + h f(t_k, y_k) \quad \text{(prediction)}$$

$$y_{k+1} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, \tilde{y}_{k+1}) \right]$$

(correction)



EXPLICIT VS IMPLICIT

Quadrature:

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))\, ds$$

$\longrightarrow$ apply a quadrature rule