Topics for today (Lecture 9) 23.5.2023

- Orthogonal polynomials
- Gauss quadrature
- Some discussion on error estimates of Gauss quadrature
- Weighted orthogonal polynomials

- Initial value problems
- Euler's method I

Orthogonal polynomials

"orthogonality" w.r.t. the inner product

given by $\langle p, q \rangle = \int_a^b p(x)\, q(x)\, dx$, p, q are real-valued polynomials on $[a, b]$.

$$\mathcal{P} := \{ p \;:\; p : [a,b] \to \mathbb{R} \text{ is polynomial} \}$$

Then $(\mathcal{P}, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space, just a (real) vector space with a positive-definite inner product.

Definition

(Inner product space) A real vector space X is called an inner product space if there exists a map

$$X \times X \longrightarrow \mathbb{R}$$

$$\langle \cdot , \cdot \rangle : X \times X \longrightarrow \mathbb{R}$$

such that

$$\langle p, q \rangle = \langle q, p \rangle \quad \forall\, p, q \in X \qquad \text{(symmetry)}$$

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall\, \alpha, \beta \in \mathbb{R},\ \forall\, x, y, z \in X \qquad \text{(bilinearity)}$$

$$\langle x, x \rangle > 0 \quad \text{iff} \quad x \neq 0. \qquad \text{(positive definiteness)}$$

Define

(norm) , any real inner product space

becomes a normed space by setting

$$\|x\| := \sqrt{\langle x, x\rangle}$$

Norm:

$$\begin{cases} \|x+y\| \le \|x\| + \|y\| & \Delta\text{-inequality} \quad \forall x, y \in X \\ \|\alpha x\| = |\alpha| \cdot \|x\| & \forall \alpha \in \mathbb{R} \ \forall x \in X \\ \|x\| = 0 \ \text{ iff } \ x = 0 & \text{(definiteness)} \end{cases}$$

iff = if and only if

Orthogonality

p, q are called orthogonal if $\langle p, q \rangle = 0$, in symbols $p \perp q$

orthonormal

if $p \perp q$ and $\|p\| = \|q\| = 1$

$= \langle p, p \rangle = \langle q, q \rangle$.

"normalized"

Finding / constructing orthogonal sequences

inner product $\langle p, q \rangle = \int_a^b p(x)\, q(x)\, dx$

norm $\| q \| = \left(\int_a^b (q(x))^2 dx \right)^{\frac{1}{2}}$

Idea is to take a basis for $\mathcal{P}$

$\{1, x, x^2, \dots, x^k, \dots\}$

Gram-Schmidt procedure

$\{q_0, q_1, \dots, q^k, \dots\}$ is orthonormal basis of $\mathcal{P}$

$$\{1, x, x^2, x^3, \ldots, \ldots\}$$

$$q_0 = \frac{1}{\|1\|} = \frac{1}{\left[\left(\int_a^b 1^2 \, dx\right)^{\frac{1}{2}}\right]} = \frac{1}{\sqrt{b-a}}$$

Identify x with the function $x \mapsto x$

for $j = 1, 2, \ldots$ (Induction!) assume that $\{q_0, q_1, \ldots, q_{j-1}\}$ has been constructed.

$$\tilde{q}_j(x) = x\, q_{j-1}(x) - \sum_{i=0}^{j-1} \langle x\, q_{j-1}, q_i \rangle\, q_i(x)$$

$$\sum_{i=0}^{j-1} \left(\int_a^b z\, q_{j-1}(z)\, q_i(z)\, dz \right) q_i(x)$$

$$q_j(x) = \frac{\tilde{q}_j(x)}{\|\tilde{q}_j\|} \qquad \left(\Longrightarrow \|q_j\| = 1 \right)$$

New basis is orthonormal!

$$\langle x q_{j-1}, q_i \rangle = \int_a^b x\, q_{j-1}(x)\, q_i(x)\, dx$$

$$\int_a^b q_{j-1}(x)\, x\, q_i(x)\, dx = \langle q_{j-1}, \underbrace{x\, q_i}_{\text{degree } i+1} \rangle$$

$$= 0 \quad \text{if}$$

$$i \leq j-3$$

$q_{j-1} \perp$ "any polynomial of degree $\leq j-2$"

$i+1 \leq j-2$

$$\tilde{q}_j(x) = x\, q_{j-1}(x) - \sum_{i=0}^{j-1} \underbrace{\langle x q_{j-1}, q_i \rangle}_{=0 \text{ if } i \leq j-3} q_i(x)$$

$$= x\, q_{j-1}(x) - \langle x q_{j-1}, q_{j-1} \rangle q_{j-1}(x)$$

$$- \langle x q_{j-1}, q_{j-2} \rangle q_{j-2}(x)$$

Works only for polynomials!

(not in Hilbert spaces)

Q: Why $\tilde{q}_j \perp \operatorname{span}\{q_0, \ldots, q_{j-1}\}$?

Observe if $\tilde{q}_j \perp \operatorname{span}\{q_0, \dots, q_{j-1}\}$

$$\Rightarrow \quad q_j \perp \operatorname{span}\{q_0, \dots, q_{j-1}\}$$

$$\langle \tilde{q}_j, q_{j-1} \rangle$$

(use bilinearity)

$$= \langle x\, q_{j-1}, q_{j-1} \rangle$$

$$- \sum_{j=0}^{j-1} \langle x\, q_{j-1}, q_i \rangle \cdot \underbrace{\langle q_i, q_{j-1} \rangle}_{=0 \text{ except for } i = j-1}$$

because $q_{j-1} \perp \operatorname{span}\{q_0, \dots, q_{j-2}\}$

$$= \langle x\, q_{j-1} , q_{j-1} \rangle$$

$$- \sum_{j=0}^{j-1} \langle x\, q_{j-1} , q_i \rangle \cdot \underbrace{\langle q_i , q_{j-1} \rangle}_{=0 \text{ except for } i=j-1}$$

$$= \langle x\, q_{j-1} , q_{j-1} \rangle - \langle x\, q_{j-1} , q_{j-1} \rangle \underbrace{\boxed{\cdot \langle q_{j-1} , q_{j-1} \rangle}}_{= \| q_{j-1} \|^2 = 1}$$

$$= 0$$

THEOREM (Gauss quadrature)

Let $x_0, x_1, x_2, \dots, x_n$ roots of an orthogonal polynomial $q_{n+1}(x)$ on $[a,b]$

Then $$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

where $$A_i = \int_a^b \varphi_i(x)\,dx, \quad \varphi_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$\int_a^b f(x)\,dx \approx \int_a^b \text{“ interpolation polynomial of } f \text{ at } x_0, x_1, \dots, x_n \text{ ”}\,dx$$

is exact for all polynomials of degree $2n+1$ or less.

1. step: Compute orthonormal polynomials on $[a,b]$

2. step: Compute roots of q_{n+1}

3. step: Compute interpolation polynomial
w.r.t. the roots $x_0, x_1, x_2, \dots, x_n$

4. step: Integrate (Exact up to degree $2n+1$)

Proof

Let f be a polynomial of degree $\leq 2n+1$

Division algorithm

$$\left(\frac{f}{q_{n+1}} = p_n + \frac{r_n}{q_{n+1}} \right)$$

$$\underset{\deg = 2n+1}{f} = \boxed{\underset{\deg = n+1}{q_{n+1}} \; \underset{\deg = n}{p_n}} + \underset{\deg \leq n}{r_n}$$

$$\frac{5}{2} = 2 + \frac{1}{2}$$

$$5 = 2 \cdot 2 + 1$$

$$f(x_i) = \underbrace{q_{n+1}(x_i)}_{=0} \, p_n(x_i) + r_n(x_i)$$ Like integer division

$$= r_n(x_i)$$

Integrale

$$\int_a^b f(x)\,dx = \underbrace{\int_a^b \overbrace{q_{n+1}(x)}^{\deg = n+1}\; \overbrace{p_n(x)}^{\deg = n}\,dx} + \int_a^b r_n(x)\,dx$$

$$= \langle q_{n+1}, p_n \rangle$$

$$= 0 \quad \text{because}$$

$$q_{n+1} \perp \operatorname{span}\{q_0, \dots, q_n\}$$

$$= \int_a^b r_n(x)\,dx = \sum_{i=0}^{n} r_n(x_i) \underbrace{\int_a^b \varphi_i(x)\,dx}_{= A_i}$$

$$= \sum_{i=0}^{n} A_i\, r_n(x_i) = \sum_{i=0}^{n} A_i\, f(x_i) \qquad \square$$

Def. Weighted orthogonal polynomials

$$\langle p, q \rangle_\omega = \int_a^b p(x)\, q(x)\, \omega(x)\, dx$$

$$\sqrt{\langle p, p \rangle_\omega} = \left(\int_a^b (p(x))^2 \omega(x)\, dx \right)^{\frac{1}{2}}$$

$\quad \shortparallel$

$\|p\|_\omega$

Typical weights $\quad e^{-x} \rightsquigarrow$ Laguerre polynomials

$e^{-\frac{x^2}{2}} \rightsquigarrow$ ~~Her~~ Hermite polynomials

Unweighted on $[-1, 1]$ $\rightsquigarrow$ Legendre polynomials
$(w = 1)$

Theorem (without proof)

The previous theorem holds for a w-orthonormal (orthogonal) polynomial q_{n+1}

$$A_i = \int_a^b \varphi_i(x)\, w(x)\, dx$$

Example (Gram-Schmidt) on $[-1,1]$

$$\tilde{q}_0 = 1$$

$$\tilde{q}_1 = x \cdot 1 - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} \cdot 1$$

$$= x - \left[\underbrace{\left(\int_{-1}^{1} x \, dx \right)}_{=0} \Big/ \left(\int_{-1}^{1} 1 \, dx \right) \right] \cdot 1$$

odd function

$$f(x) = -f(-x)$$

$$\int_{-1}^{1} \text{"odd function" } dx = 0$$

$$= x$$

$$q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \frac{x}{\left(\int_{-1}^{1} x^2 \, dx\right)^{\frac{1}{2}}} = \frac{x}{\sqrt{\frac{x^3}{3}\Big|_{-1}^{1}}}$$

$$\|q_1\|^2 = \frac{2}{3}$$

$$= \sqrt{\frac{3}{2}} \; x$$

$$\langle x, x\rangle = \frac{2}{3}$$

odd function

$$\tilde{q}_2 = x \cdot x - \frac{\langle x^2, 1\rangle}{\underbrace{\langle 1, 1\rangle}_{=2}} \, 1 - \frac{\langle x^2, x\rangle = 0}{\underbrace{\langle x, x\rangle}_{=\frac{2}{3}}} \cdot x$$

q_0 $\quad$ q_1

$$= x^2 - \frac{\frac{2}{3}}{2} \cdot 1 = x^2 - \frac{1}{3}$$

$$\tilde{q}_2 = x^2 - \frac{1}{3}$$ Roots $x_0 = -\frac{1}{\sqrt{3}}$, $x_1 = \frac{1}{\sqrt{3}}$

(Legendre polynomials)

n=1 → 2 roots

Quadrature

$$\int_{-1}^{1} f(x)\,dx = A_0 f\left(-\frac{1}{\sqrt{3}}\right) + A_1 f\left(\frac{1}{\sqrt{3}}\right)$$

by the way

$$\tilde{q}_3 = x^3 - \frac{3}{5}x$$

Now $\int_{-1}^{1} 1\,dx = 2 = A_0 + A_1$

$\int_{-1}^{1} x\,dx = 0 = -\frac{A_0}{\sqrt{3}} + \frac{A_1}{\sqrt{3}}$

$\Rightarrow A_0 = A_1 = 1$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = 1 \cdot \left(-\frac{1}{\sqrt{3}}\right)^2 + 1\left(\frac{1}{\sqrt{3}}\right)^2$$

$$\int_{-1}^{1} x^3\,dx = 0 = 1\left(-\frac{1}{\sqrt{3}}\right)^3 + 1\left(\frac{1}{\sqrt{3}}\right)^3$$

$\rightarrow$ Gauss Quadrature is exact for degree $2n+1 = 3 = 2 \cdot 1 + 1$