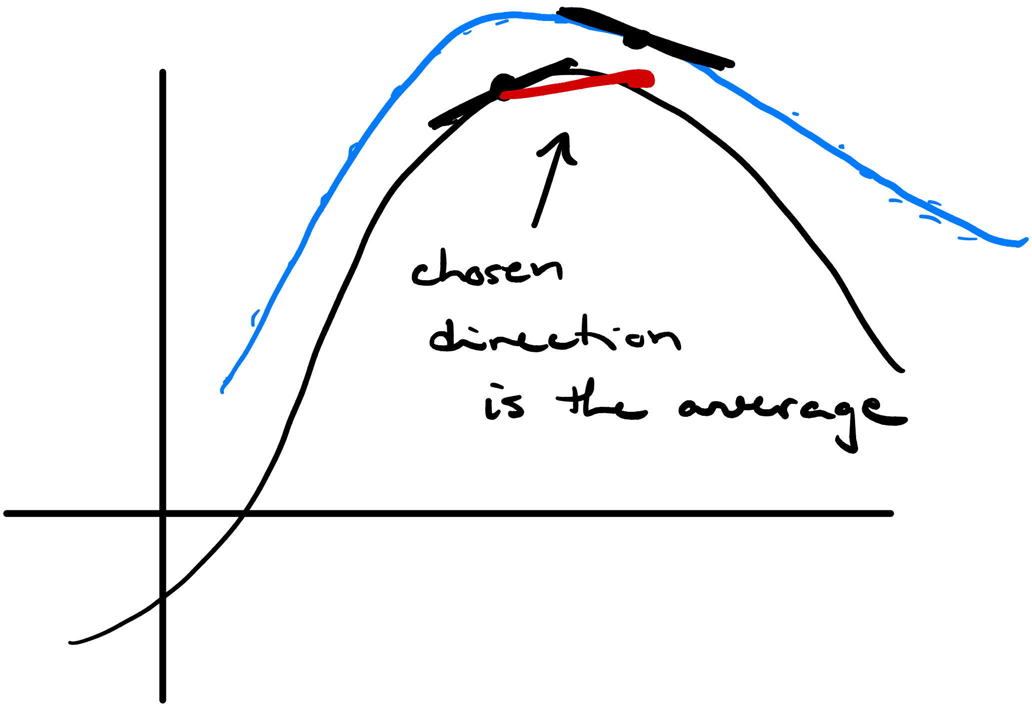
Idea: Predictor - corrector

Heun's Method

$$\tilde{y}_{k+1} = y_k + h f(t_k, y_k) \quad \text{(prediction)}$$

$$y_{k+1} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, \tilde{y}_{k+1}) \right]$$

(correction)



EXPLICIT vs IMPLICIT

Quadrature:

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))\, ds$$

→ apply a quadrature rule

for instance $\frac{h}{2}\left[f(t,y(t)) + f(t+h, y(t+h))\right]$

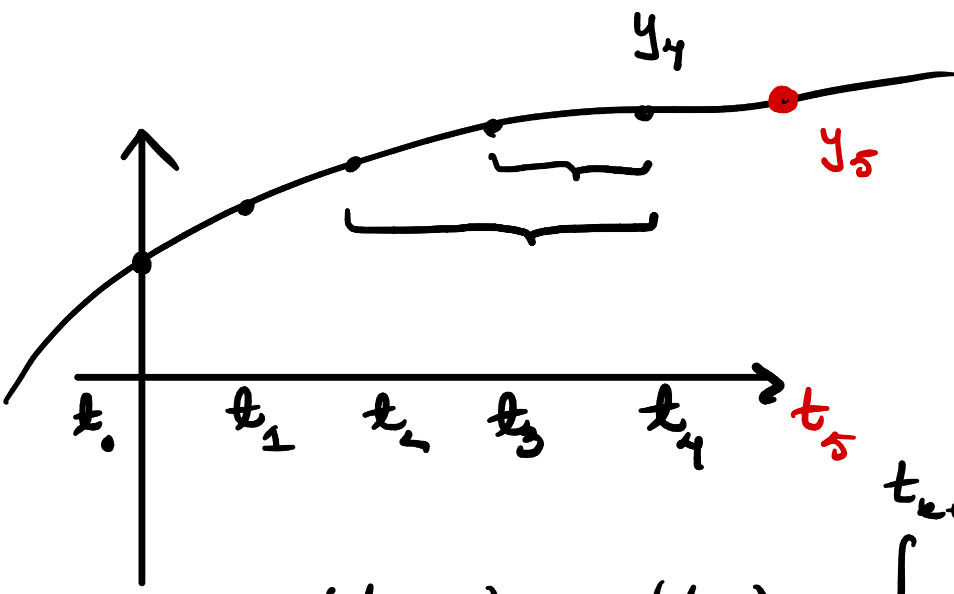
$+ \Theta(h^3)$

Combined we get:

$$\underline{\underline{y_{k+1}}} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, \underline{\underline{y_{k+1}}})\right]$$

This method is implicit. Every step requires a solution of a nonlinear equation.

MULTI-STEP METHODS



$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(s, y(s))\, ds$$

Adams – Bashforth (Explicit)

Interpolation nodes : $t_k, t_{k-1}, \dots, t_{k-m+1}$

Polynomial : $P_{m-1}(s)$

$$y_{k+1} = y_k + \int_{t_k}^{t_{k+1}} P_{m-1}(s)\, ds$$

$$= y_k + h \sum_{l=0}^{m-1} b_l\, f(t_{k-l}, y_{k-l})$$

where $$b_l = \frac{1}{h} \int_{t_k}^{t_{k+1}} \left(\prod_{\substack{j=0 \\ j\neq l}}^{m-1} \frac{s - t_{k-j}}{t_{k-l} - t_{k-j}} \right) ds$$

What methods can be recovered ?

Set $m = 1$: $l = 0 \Rightarrow b_0 = 1$

We get
$$y_{k+1} = y_k + h f(t_k, y_k)$$

Euler's method !

Truncation error is $O(h^m)$.

Adams - Moulton (Implicit)

Add t_{k+1} ; $q_m(s)$; Truncation error $O(h^{m+1})$

$$y_{k+1} = y_k + h \sum_{l=0}^{m} c_l f(t_{k+1-l}, y_{k+1-l})$$

where
$$c_l = \frac{1}{h} \int_{t_k}^{t_{k+1}} \left(\prod_{\substack{j=0 \\ j \neq l}} \frac{s - t_{k+1-j}}{t_{k+1-l} - t_{k+1-j}} \right) ds$$

Let $m=0$: $l=0$; $c_0 = 1$

We get $y_{k+1} = y_k + h f(t_{k+1}, y_{k+1})$

Backward Euler!

General form:

$$\sum_{l=0}^{m} a_l y_l = h \sum_{l=0}^{m} b_l f(t_{k+l}, y_{k+l})$$

Normalise: $a_m = 1$

If $b_m = 0 \implies$ explicit

otherwise $\implies$ implicit

EXAMPLE (GOOD BAD EXAMPLE)

$$y_{k+2} - 3y_{k+1} + 2y_k =$$

$$h\Big[\frac{13}{12} f(t_{k+2}, y_{k+2})$$

$$- \frac{5}{3} f(t_{k+1}, y_{k+1})$$

$$- \frac{5}{12} f(t_k, y_k) \Big]$$

<u>Solve</u> : $y' = 0 \,, \quad y(0) = 1$

Let us introduce an error to the initial values :

$$y_0 = 1 \,, \quad y_1 = 1 + \delta$$

$$y_2 = 3y_1 - 2y_k = 1 + 3\delta$$

...

$$y_k = 3y_{k-1} - 2y_k = 1 + (2^k - 1)\delta$$

$\delta \sim 2^{-53}$ $\Rightarrow k = 100$

$\Rightarrow$ error $\sim 2^{47}$

EXAMPLE (Effect of rounding error)

Euler: Return to the proof:

$$|d_{k+1}| \leq (1 + hL)\,|d_k| + \delta$$

$$\Rightarrow \quad |d_{k+1}| \leq e^{L(T-t_0)}\,|d_0|$$

initial error or uncertainty

$$+ \underbrace{\frac{e^{L(T-t_0)} - 1}{hL}\,\delta}_{\text{dominant term, if } h \text{ is sufficiently small}}$$

Theorem The order of the truncation error of a multi-step method is $p \geq 1$, if and only if

$$\sum_{l=0}^{m} a_l = 0$$

$$\sum_{l=0}^{m} l^j a_l = j \sum_{l=0}^{m} l^{j-1} b_l \, , \quad j = 1, \dots, p.$$

EXAMPLE $m = 1$:

$$a_0 + a_1 = 0$$

$$0 \cdot a_0 + 1 a_1 = b_1$$

Normalise : $a_1 = 1 \Rightarrow a_0 = -1$

$$\Rightarrow b_1 = 1$$

We get :

$$y_{k+1} = y_k + h f(t_{k+1}, y_{k+1})$$

Backward Euler !