Demouppgifter 6

① Låt F vara vektorfältet $F(R,\theta,z) = R\hat{R}$ skrivet i cylindriska koordinater. Beräkna $\operatorname{div} F$ och $\operatorname{Curl} F$.

$\vec{r}(R,\theta,z) =$
$= (R\cos\theta, R\sin\theta, z)$

Lösning: Kom ihåg,
$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases} \Longrightarrow \begin{cases} \hat{R} = (\cos\theta, \sin\theta, 0) \\ \hat{\theta} = (-\sin\theta, \cos\theta, 0) \\ \hat{z} = (0,0,1) \end{cases}$$

Dessutom $h_R = 1$, $h_\theta = R$, $h_z = 1$

(Eftersom $h_R = \left|\frac{\partial \vec{r}}{\partial R}\right| = 1$, $h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = R$, $h_z = \left|\frac{\partial \vec{r}}{\partial z}\right| = 1$.)

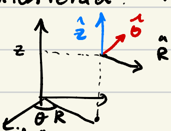
Vi vet att $\operatorname{div} \vec{F} = \frac{1}{h_R h_\theta h_z}\left(\frac{\partial}{\partial R}(F_R h_\theta h_z) + \frac{\partial}{\partial \theta}(h_R F_\theta h_z) + \frac{\partial}{\partial z}(h_R h_\theta F_z)\right) =$

Detta ger, eftersom $F_R = R$, $F_\theta = F_z = 0$,

$$\operatorname{div} F = \frac{1}{R}\frac{\partial}{\partial R}(R^2) = \frac{1}{R}\cdot 2R = 2$$

[Man kan notera att $F = R\hat{R}$ är detsamma som $F(x,y,z) = (x,y,0)$ så $\operatorname{div} F = 2$ är korrekt.]

Nu, $\operatorname{Curl} F$. För detta behöver vi ordna basen så den blir högerorienterad. Vi kollar lätt att $[\hat{R}, \hat{\theta}, \hat{z}]$ är det.



Därför $\operatorname{Curl} F = \frac{1}{h_R h_\theta h_z}\begin{vmatrix} h_R\hat{R} & h_\theta\hat{\theta} & h_z\hat{z} \\ \partial/\partial R & \partial/\partial\theta & \partial/\partial z \\ F_R h_R & F_\theta h_\theta & F_z h_z \end{vmatrix} =$

$$= \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & \hat{z} \\ \partial/\partial R & \partial/\partial\theta & \partial/\partial z \\ R & 0 & 0 \end{vmatrix} = \frac{1}{R}\left(\begin{vmatrix} \partial/\partial\theta & \partial/\partial z \\ 0 & 0 \end{vmatrix}\hat{R} - \begin{vmatrix} \partial/\partial R & \partial/\partial z \\ R & 0 \end{vmatrix}R\hat{\theta} + \begin{vmatrix} \partial/\partial R & \partial/\partial\theta \\ R & 0 \end{vmatrix}\hat{z}\right) = \vec{0}.$$

(2) Låt $F(R,\theta)$ vara ett vektorfält skrivet i polära koordinater. Härled en formel för div F.

Lösning: Vi har $F = F_R \hat{R} + F_\theta \hat{\theta}$ och vet att $h_R = 1$ & $h_\theta = R$.

Därför
$$\operatorname{div} \vec{F} = \frac{1}{h_R h_\theta}\left(\frac{\partial}{\partial R}(F_R h_\theta) + \frac{\partial}{\partial \theta}(h_R F_\theta)\right) =$$
$$= \frac{1}{R}\left(\frac{\partial}{\partial R}(R F_R) + \frac{\partial}{\partial \theta} F_\theta\right) =$$
$$= \frac{1}{R}\left(F_R + R\frac{\partial F_R}{\partial R} + \frac{\partial F_\theta}{\partial \theta}\right) = \frac{\partial F_R}{\partial R} + \frac{1}{R}\left(F_R + \frac{\partial F_\theta}{\partial \theta}\right)$$

(3) Låt $f(R,\theta,z)$ vara en funktion skriven i cylindriska koordinater. Laplacianen av f är definierad som $\Delta f = \operatorname{div}(\nabla f)$. Härled en formel för Δf i cylindriska koordinater.

Lösning: Vi vet (se övning 1) att $h_R = h_z = 1$ & $h_\theta = R$.

Vi vet också att
$$\nabla f = \frac{1}{h_R}\frac{\partial f}{\partial R}\hat{R} + \frac{1}{h_\theta}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{h_z}\frac{\partial f}{\partial z}\hat{z} =$$
$$= \frac{\partial f}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$$

Med hjälp av formeln för divergens får vi
$$\operatorname{div}(\nabla f) = \operatorname{div}\left(\frac{\partial f}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}\right) =$$
$$= \frac{1}{h_R h_\theta h_z}\left(\frac{\partial}{\partial R}\left(h_\theta h_z \frac{\partial f}{\partial R}\right) + \frac{\partial}{\partial \theta}\left(h_R h_z \cdot \frac{1}{R}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(h_R h_\theta \frac{\partial f}{\partial z}\right)\right)$$
$$= \frac{1}{R}\left(\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{R}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(R\frac{\partial f}{\partial z}\right)\right) =$$
$$= \frac{1}{R}\left(\frac{\partial f}{\partial R} + R\frac{\partial^2 f}{\partial R^2} + \frac{1}{R}\frac{\partial^2 f}{\partial \theta^2} + R\frac{\partial^2 f}{\partial z^2}\right) = \frac{1}{R}\frac{\partial f}{\partial R} + \frac{\partial^2 f}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

Inlämningsuppgift 6

(1) Låt $F(x,y) = \left(-y + x\sqrt{x^2+y^2},\ x + y\sqrt{x^2+y^2}\right)$

a) Skriv vektorfältet i polära koordinater, med andra ord, bestäm F_R och F_θ i
$F = F_R\hat{R} + F_\theta\hat{\theta}$

b) Beräkna div F uttryckt i polära koordinater.

Lösning: I polära koordinater $\begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases}$

Dessutom $\hat{R} = (\cos\theta, \sin\theta)$ och $\hat{\theta} = (-\sin\theta, \cos\theta)$.

Därför

$$F_R = F\cdot\hat{r} = \left(-y + x\sqrt{x^2+y^2}\right)\cos\theta + \left(x + y\sqrt{x^2+y^2}\right)\sin\theta =$$

$$= \left(-R\sin\theta + R^2\cos\theta\right)\cos\theta + \left(R\cos\theta + R^2\sin\theta\right)\sin\theta =$$

$$= R^2\cos^2\theta + R^2\sin^2\theta = R^2$$

och

$$F_\theta = F\cdot\hat{\theta} = \left(-y + x\sqrt{x^2+y^2}\right)(-\sin\theta) + \left(x + y\sqrt{x^2+y^2}\right)\cos\theta =$$

$$= \left(-R\sin\theta + R^2\cos\theta\right)(-\sin\theta) + \left(R\cos\theta + R^2\sin\theta\right)\cos\theta =$$

$$= R\sin^2\theta + R\cos^2\theta = R$$

$$\Rightarrow F = R^2\hat{R} + R\hat{\theta}$$

b) Vi har $h_R = 1$ och $h_\theta = R$. Dessutom

$$\operatorname{div} F = \frac{1}{h_R h_\theta}\left(\frac{\partial}{\partial R}(F_R h_\theta) + \frac{\partial}{\partial \theta}(h_R F_\theta)\right) =$$

$$= \frac{1}{R}\left(\frac{\partial}{\partial R}(R^2 \cdot R) + \frac{\partial}{\partial \theta}(1 \cdot R)\right) =$$

$$= \frac{1}{R} \cdot 3R^2 = 3R$$

② Definiera kroklinjära koordinater i xy-planet via

$$\vec{r}(u,v) = (u^2 - v^2, 2uv) = (x(u,v), y(u,v))$$

a) Visa att detta kroklinjära koordinatsystem är ortogonalt då $(x,y) \neq (0,0)$

b) Beräkna skalfaktorerna h_u & h_v.

Lösning Vi beräknar $\frac{\partial \vec{r}}{\partial u} = (2u, 2v)$ och $\frac{\partial \vec{r}}{\partial v} = (-2v, 2u)$. Vi vet att

$$\frac{\partial \vec{r}}{\partial u} = \left|\frac{\partial r}{\partial u}\right| \hat{u} = h_u \hat{u} \quad \text{och} \quad \frac{\partial \vec{r}}{\partial v} = h_v \hat{v}$$

$\frac{\partial \vec{r}}{\partial u} \neq (0,0)$ och $\frac{\partial \vec{r}}{\partial v} \neq (0,0)$ om $(u,v) \neq (0,0)$ $\Rightarrow (x,y) \neq (0,0)$

Om $\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = 0$ så är också $\hat{u} \cdot \hat{v} = 0$

Notera att $\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = 2u \cdot (-2v) + 2v \cdot 2u = 0$

$\Longrightarrow$ Koordinatsystemet är ortogonalt då $(x,y) \neq (0,0)$.

b) Beräkna längderna av $\frac{\partial \vec{r}}{\partial u}$ och $\frac{\partial \vec{r}}{\partial v}$

$$h_u = \left|\frac{\partial \vec{r}}{\partial u}\right| = \sqrt{(2u)^2+(-2v)^2} = 2\sqrt{u^2+v^2}$$

$$h_v = \left|\frac{\partial \vec{r}}{\partial v}\right| = \sqrt{(2u)^2+(2v)^2} = 2\sqrt{u^2+v^2}$$

③ Låt $\vec{F}(R,\theta,z) = R^2\hat{R} + R\hat{\theta} + z\hat{z}$ vara ett vektorfält skrivet i cylindriska koordinater.

a) Beräkna $\operatorname{div}\vec{F}$

b) Beräkna $\operatorname{Curl}\vec{F}$

Lösning: Vi vet att $h_R=1$, $h_\theta = R$ och $h_z = 1$ för cylindriska koordinater.

a) Därför

$$\operatorname{div}\vec{F} = \frac{1}{h_R h_\theta h_z}\left(\frac{\partial}{\partial r}(F_R h_\theta h_z) + \frac{\partial}{\partial \theta}(h_R F_\theta h_z) + \frac{\partial}{\partial z}(h_R h_\theta F_z)\right)$$

$$= \frac{1}{R}\left(\frac{\partial}{\partial R}(R^2 \cdot R) + \frac{\partial}{\partial \theta}(R) + \frac{\partial}{\partial z}(Rz)\right) =$$

$$= \frac{1}{R}(3R^2 + 0 + R) = 3R+1.$$

b) Enligt formel

$$\operatorname{Curl}\vec{F} = \frac{1}{h_R h_\theta h_z}\begin{vmatrix} h_R\hat{R} & h_\theta\hat{\theta} & h_z\hat{z} \\ \partial/\partial R & \partial/\partial\theta & \partial/\partial z \\ F_R h_R & F_\theta h_\theta & F_z h_z \end{vmatrix} =$$

$$= \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & \hat{z} \\ \partial/\partial R & \partial/\partial\theta & \partial/\partial z \\ R^2 & R^2 & z \end{vmatrix} =$$

$$= \frac{1}{R}\left(\begin{vmatrix} \partial/\partial\theta & \partial/\partial z \\ R^2 & z \end{vmatrix}\hat{R} - \begin{vmatrix} \partial/\partial R & \partial/\partial z \\ R^2 & z \end{vmatrix} R\hat{\theta} + \begin{vmatrix} \partial/\partial r & \partial/\partial\theta \\ R^2 & R^2 \end{vmatrix}\hat{z}\right) =$$

$$= \frac{1}{R}\left(0\hat{R} - 0R\hat{\theta} + 2R\hat{z}\right) = 2\hat{z}.$$

(4) Låt $f(R,\phi,\theta)$ vara en funktion skriven i sfäriska koordinater. Härled en formel för

$$\Delta f = \operatorname{div}(\nabla f)$$

i sfäriska koordinater.

Lösning: Skalfaktorerna är

$$h_R = 1\ ,\ h_\phi = R \quad \text{och} \quad h_\theta = R\sin\phi$$

Därför

$$\nabla f = \frac{1}{h_R}\frac{\partial f}{\partial R}\hat{R} + \frac{1}{h_\phi}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{1}{h_\theta}\frac{\partial f}{\partial\theta}\hat{\theta} =$$

$$= \frac{\partial f}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{1}{R\sin\phi}\frac{\partial f}{\partial\theta}\hat{\theta}$$

Slutligen

$$\Delta f = \operatorname{div}(\nabla f) = \frac{1}{R^2\sin\phi}\left(\frac{\partial}{\partial R}\left(R^2\sin\phi\frac{\partial f}{\partial R}\right) + \right.$$

$$\left. + \frac{\partial}{\partial\phi}\left(R\sin\phi\,\frac{1}{R}\frac{\partial f}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{R}{R\sin\phi}\frac{\partial f}{\partial\theta}\right)\right) =$$

$$= \frac{1}{R^2\sin\phi}\left(2R\sin\phi\frac{\partial f}{\partial R} + R^2\sin\phi\frac{\partial^2 f}{\partial R^2} + \cos\phi\frac{\partial f}{\partial\phi} + \sin\phi\frac{\partial^2 f}{\partial\phi^2} + \right.$$

$$\left. + \frac{1}{\sin\phi}\frac{\partial^2 f}{\partial\theta^2}\right) =$$

$$= \frac{2}{R}\frac{\partial f}{\partial R} + \frac{\partial^2 f}{\partial R^2} + \frac{\cos\phi}{R^2\sin\phi}\frac{\partial f}{\partial\phi} + \frac{1}{R^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}\Big)$$