Topic P02: Introduction to the Bell inequalities

Tuomas Juuranto

The underlying principle of physics has, for most of human history, been determinism: if one could measure everything about a certain system, one should be able to predict with certainty what occurs next in that system. However, the paradigm of determinism started to crumble in the early 20th century when the field of quantum mechanics began to form. The probabilistic nature of quantum mechanics differed greatly from the nature of classical physics, which led to scepticism among physicists. Many of these physicists, including Albert Einstein, thought that quantum mechanics was an incomplete theory. They believed that there existed some “hidden variable” of which we were still unaware of, and which could reinstate determinism. One of the most famous thought experiments in favour of hidden variables was the Einstein–Podolsky–Rosen (EPR) paradox [1]. However, the physicist John Stewart Bell showed in his reply to the EPR paradox that the idea of hidden variables presented in this paradox was incompatible with the statistical predictions of quantum mechanics [2]. From this reply arose the Bell’s theorem and Bell inequalities, which we will examine more closely in this essay.

Bell inequalities are a certain type of inequalities that must be obeyed by any local realist theory. In other words, any question with a definite answer must obey the Bell inequalities. Here, the term “local” refers to the concept that superluminal signalling is impossible, and the term “realist” refers to the idea that the outcome of an experiment is entirely determined by the system’s properties that exist even if the system is not examined. In Bell’s reply to the EPR paradox, he constructed the first Bell inequality by assuming that there existed some hidden-variable λ, and showed that a certain quantum mechanics experiment violated this inequality [2]. However, the experiment constructed by Bell could not be actually constructed in real life due to the assumptions made about the detectors. This challenge was overcome in 1969 when J.Clauser, M. Horne, A. Shimony and R. Holt proposed a variation on the Bell inequality that could be actually checked by an experiment [3]. This specific Bell inequality became known as the CHSH inequality.

In CHSH inequality, a pair of entangled particles is created, both of which have a spin of one half [4]. Entanglement in quantum mechanics implies that the particles are connected, and measuring a property of one particle also determines the property of the other particle. The spin of one particle is measured by Alice and the spin of the other by Bob[1]. Alice can measure the spin in two different directions but only in one direction at once. Let us denote these measurements of spin in different directions as a_1 and a_2. Likewise, Bob can perform the measurement of spin in two directions denoted as b_1 and b_2. Assuming local realism, each of these measurements should have a definite answer even if the measurement is not performed. Let us denote the results of these measurements as A_1 ,A_2 ,B_1 and B_2 , and let these results correspond to discrete values of ±1. With these assumptions, we obtain an equality

$$A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 = A_1(B_1 + B_2) + A_2(B_1 - B_2) = \pm 2 \tag{1}$$

This equality holds, since one of the terms $B_1 + B_2$ and $B_1 - B_2$ is always 0 and the other ±2 regardless of the results, and results A_1 and A_2 both take values of ±1. Next, let us repeat this experiment numerous times. In each one, Alice chooses either measurement a_1 or a_2 and Bob chooses either b_1 or b_2. Since equation (1) holds for every individual experiment, we obtain

$$S = |E(a_1, b_1) + E(a_1, b_2) + E(a_2, b_1) - E(a_2, b_2)| < 2 \tag{2}$$

, where S denotes the absolute sum itself and $E(a_1, b_1)$ denotes the average of the outcomes calculated by equation (1) in measurements, where Alice chose a_1 and Bob chose b_1, etc. This inequality holds for any realist theory [4].

Let us next compare this inequality (2) to the predictions of quantum theory. The spin wave function of an entangled individual particle is the so-called Bell type, which

[1] Alice and Bob are the typical names of the entities in many quantum mechanical experiments

means it is a superposition of the two possible states [4]. From this Bell type wave function, one can choose the two measurement directions so that

$$E(a_1, b_1) = E(a_1, b_2) = E(a_2, b_1) =- E(a_2, b_2) = \frac{1}{\sqrt{2}} \qquad (3)$$

From equality (3), it follows that $S = 2\sqrt{2}$, which clearly violates the inequality (2).

The first Bell inequality, the CHSH inequality and numerous other results together form Bell's theorem, which states that local hidden-variable theories and quantum mechanics are incompatible. In Bell’s own words “If [a hidden-variable theory] is local it will not agree with quantum mechanics, and if it agrees with quantum mechanics it will not be local [5].”

To summarise, Bell inequalities are a certain type of inequalities which local realist theories must obey. However, quantum mechanics disobeys these inequalities, from which one can conclude that quantum mechanics is not simultaneously compatible with the principles of locality and realism [2,5]. Many scientific experiments testing the locality and realism of quantum mechanics have been conducted since J.S. Bell’s reply in 1964. These experiments have managed to close most of the conceivable loopholes related to Bell’s theorem except some of the more extraordinary ones, such as superdeterminism, which questions the free will of humans [6]. Despite these extraordinary loopholes, the current scientific consensus agrees on the probabilistic nature of quantum mechanics.

References

[1] A. Einsteins, B. Podolsky and N. Rosen, “Can Quantum-Mechanical Description of Physical Reality be Considered Complete?”, *Phys. Rev*. vol. 47, no. 8, pp. 777-780. May 1935.

[2] J.S. Bell. “On Einstein Podolsky Rosen Paradox”, *Physics Physique Физика*. vol. 1, no. 3, pp. 195-200. Nov. 1964.

[3] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, “Proposed experiment to test local hidden-variable theories”, *Physical Review Letter*s. vol. 23, no. 15, pp. 880-884. Oct. 1969.

[4] The Nobel Committee for Physics, ““For experiments with entangled photons, establishing the violation of bell inequalities and pioneering quantum information science”, “ *nobelprize.org*, Oct. 04, 2022.

[5] J.S. Bell. *Speakable and Unspeakable in Quantum Mechanics*. Cambridge: Cambridge University Press, 1987, p. 65

[6] B.Hensen, et al. “Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres”, *Nature*. Vol 526, issue 7575, pp.682-686. Oct. 2015.