<u>C Level Questions</u>

1.      Consider the Solow Growth Model which has technological progress and population growth. The economy is described by:

$$y = k^{.3}$$
$$s = .1$$
$$\delta = .03$$
$$n = .02$$
$$g = .02$$

a.  Solve for the steady state level of capital per capita and output per capita.  In the steady state, how fast does capital per capita grow?  How fast does output per capita grow?  How fast does total output grow?

<span style="color:red">This was a mis-worded question.  I should have asked for the steady state level of capital per efficient worker and output per efficient worker.  The steady state level of capital per efficient worker occurs when $.1k^{.3} = .07k$ or when k = 1.664.  Output per efficient worker iis $1.664^{.3} = 1.165$. Total output will grow at 4% per period and output per capita will grow 2%.</span>

b.  15 years ago (1991), American Real GDP was $6,720.9 billion and the labor force was 117,770 thousand people.  10 years later (2001), American Real GDP was $9348.6 billion and the labor force was 134,253 thousand people.  Using these numbers and assuming the Solow Growth model is correct, determine the average annual technological growth rate for the United States over the last ten years.

<span style="color:red">If the Solow growth model is correct, then output per person will grow each period at a rate of g. From the above data, output per person in 1991 was $57,068.014 and output per person in 2001 was 69,634.198.  Over ten years this was a 22% growth which translates into a per period growth rate of : $69,634.198 = 57,068.014 (1+ x)^{10}$ so x = .0201.  The annual growth rate of per capita real GDP is 2.01%.  This is our best guess at g.</span>

2.      The purpose of this problem is to simulate the Solow Growth model using Excel (or a similar spreadsheet).  At the completion of this problem, you should be able to identify steady state levels of growth per capita, the speed of economic growth, and how the per capita variables translate into the total production, labor, and capital in an economy.

For this entire homework, you will use the following equations:

$$Y = K^{\frac{1}{3}}L^{\frac{2}{3}}$$
$$y = k^{\frac{1}{3}}$$
$$savings = s\,k^{\frac{1}{3}}$$
$$depreciation = \delta k$$

In the Solow Growth chapter, Mankiw estimates that for the United States, $\delta$=.04 and the average savings rate in the U.S. over the last 30 years is s=.068.  Using these values, create a spreadsheet that runs for 400 periods that contains the following information in period 1.

| Period | y | k | savings | depreciation | c | Y | K | L |
|--------|---|------|---------|--------------|---|---|---|-----|
| 1 | | 1.15 | | | | | | 100 |
| 2 | | | | | | | | 100 |

A few notes:

A.  We begin by assuming that capital per person is 1.15.  This will grow over time based upon the difference between savings and depreciation.

B.  We also assume that there are 100 people in our economy for each year; this will not change until you are asked to change it.

C.  You will need to fill in the rest of the blanks with formulas that compute the relevant numbers and copy down for 400 periods.  The best way to do this is to compute the per capita variables according to the equations in Chapter 4 and then compute the variables Y and K by remembering Y=L*y and K=L*k.  After you've succesfully done this for the first and second year, you should be able to use the "copy down" feature in Excel to paste your new equations in the remaining time periods.

Questions:

a.  Given s=.068 and δ=.04, mathematically find the steady state level of k and y (this does not require Excel—as a matter of fact, you should attempt this before running any Excel program). The steady state occurs where total savings is equal to total depreciation.  In this case, that is when $.04k = .068k^{(1/3)}$.  Solving for k gives $k = (.068/.04)^{1.5} = 2.2165$.  When the economy has 2.2165 units of capital, its steady state level of output is $2.2165^{(1/3)} = 1.3038$.

b.  After 400 periods, have the values of y and k reached their steady state levels?  Why or why not? Plot and print the values of y and Y to help answer this question.
The first few periods I calculate are:

| Period | y | k | savings | depreciation | c | Y | K | L |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.04769 | 1.15 | 0.071243 | 0.046 | 0.976447 | 104.769 | 115 | 100 |
| 2 | 1.0553 | 1.175243 | 0.07176 | 0.0470097 | 0.983539 | 105.53 | 117.5243 | 100 |
| 3 | 1.062657 | 1.199994 | 0.072261 | 0.0479997 | 0.990396 | 106.2657 | 119.9994 | 100 |
| 4 | 1.06977 | 1.224254 | 0.072744 | 0.0489702 | 0.997026 | 106.977 | 122.4254 | 100 |

The last few periods are:

| Period | y | k | savings | depreciation | c | Y | K | L |
|---|---|---|---|---|---|---|---|---|
| 396 | 1.303835 | 2.216502 | 0.088661 | 0.0886601 | 1.215174 | 130.3835 | 221.6502 | 100 |
| 397 | 1.303835 | 2.216502 | 0.088661 | 0.0886601 | 1.215174 | 130.3835 | 221.6502 | 100 |
| 398 | 1.303835 | 2.216503 | 0.088661 | 0.0886601 | 1.215175 | 130.3835 | 221.6503 | 100 |
| 399 | 1.303836 | 2.216504 | 0.088661 | 0.0886601 | 1.215175 | 130.3836 | 221.6504 | 100 |
| 400 | 1.303836 | 2.216504 | 0.088661 | 0.0886602 | 1.215175 | 130.3836 | 221.6504 | 100 |

For all practical purposes, we have reached a steady state by period 400.  Notice that the values k and y in period 400 are equal to those solved for in problem a.  Technically though (and not practically), we still have yet to reach the steady state.  This is an example of Xeno's paradox—we are continuously moving a fraction closer to the steady state but, even if I move half the distance each period, there is still a little distance remaining to be moved at the end of each period.

To show that we have reached the steady state for all practical purposes, consider my plots of y and Y:

y



Y
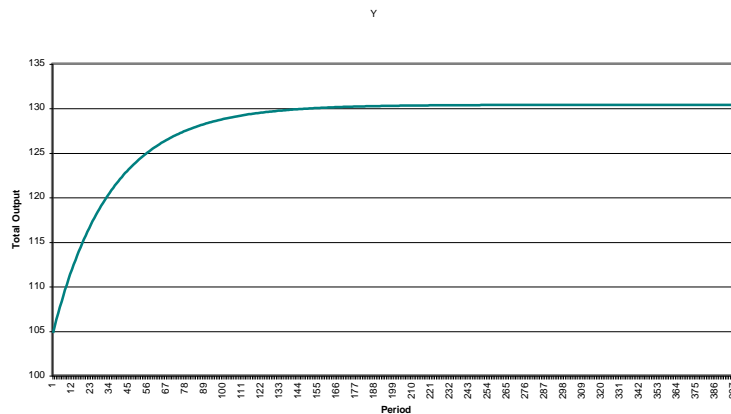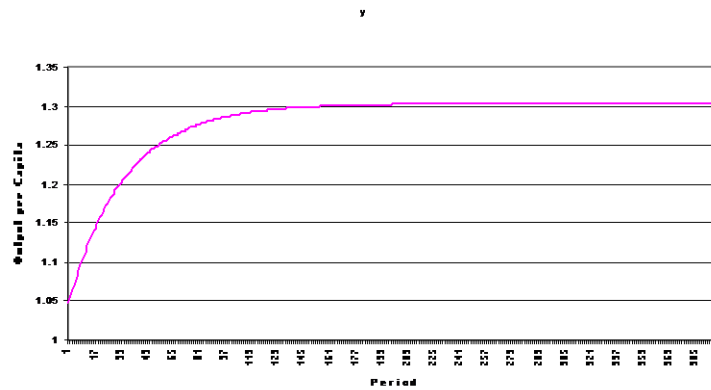
c. What is the growth rate of total output between period 1 and period 400?  Compare this to the growth rate between two periods (1 and 10) and (391 and 400).  Which subperiod grows faster? Why?  Is the growth rate of total output different than the growth rate of per capita output?  Why or why not?

$Y_1 = 104.769$, $Y_{10} = 110.7861$, $Y_{391} = 130.3834359$, $Y_{400} = 130.3835681$.  The per period growth rate for the first subset is the x that solves:

$104.76(1 + x)^9 = 110.7861$.

Using logs, I find x = .0062.  In other words, the average per period growth rate in y over the first ten periods is .62%.  Likewise, the average per period growth rate in y for the last ten periods is .0000112%.

The first subperiod grows faster than the last because of diminishing returns to capital.  During the first period, the capital stock is relatively low so adding additional capital will increase production quickly.  As the gap between savings and depreciation is large, we add a lot of capital to an economy that can do a lot with it and thus we get large production growth.  During the last periods the economy has a large amount of capital, hence the marginal productivity of capital is low, so adding capital doesn't increase output by much.  Also, at the end of the time period little net additional capital is added since depreciation is almost equal to savings.

d. Now imagine that each period, the labor force grows by 2% (n = .02).  Mathematically solve for the steady state level of capital per person and output per person.

Here we solve for a new steady state level of capital that includes capital per person declining due to inflation and population growth.  In class we saw the steady state occurs when $(n + \delta)k = sk^{(1/3)}$.

Again we solve for k and in general get $k = (s/(n + \delta))^{(3/2)}$. In this case s = .068, n = .02, $\delta$ = .04 so our steady state level of capital is $(.068/.06)^{1.5} = 1.2065$. The steady state level of output per person is simply $1.2065^{(1/3)} = 1.06458$.

e. Produce another computer model similar to the one above including the growth in labor force. How much do y and Y grow in the steady state? Plot and print both y and Y over time. Does this match what we observe in the United States?
This set-up is a little more difficult. Here are the first three periods worth of data that I constructed:

| Period | y | k | savings | depreciation | c | Y | K | L |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.04769 | 1.15 | 0.071243 | 0.046 | 0.976447 | 104.7689553 | 115 | 100 |
| 2 | 1.048357 | 1.152199 | 0.071288 | 0.046088 | 0.977069 | 106.9324027 | 117.5243 | 102 |
| 3 | 1.048998 | 1.154313 | 0.071332 | 0.0461725 | 0.977666 | 109.1377178 | 120.0947 | 104.04 |

Here's my excel work that I used to construct that data:

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | Period | y | k | savings | depreciation | c | Y | K | L |
| 2 | 1 | =C2^(1/3) | =H2/I2 | =0.068*B2 | =0.04*C2 | =B2-D2 | =I2*B2 | 115 | 100 |
| 3 | =A2+1 | =C3^(1/3) | =(H2+D2*I2 -E2*I2)/I3 | =0.068*B3 | =0.04*C3 | =B3-D3 | =I3*B3 | =I3*C3 | =I2*1.02 |
| 4 | =A3+1 | =C4^(1/3) | =(H3+D3*I3 -E3*I3)/I4 | =0.068*B4 | =0.04*C4 | =B4-D4 | =I4*B4 | =I4*C4 | =I3*1.02 |

My plot of output per capita looks similar as before (although it reaches a lower level of steady state).

3. [This question is meant to address the assumption that "per capita" refers to per worker variables rather than per population variables]

Consider the standard Solow Growth model where output is given by $Y = K^{\frac{1}{2}}L^{\frac{1}{2}}$. However, the population of this economy is given by P. Assume that a constant percentage, $\psi$, of the population chooses not to participate in the labor force (so $L = (1 - \psi)P$).

a. Solve for the per-population production function (I'll denote this $y$ as opposed to y which will remain the per worker production function). Compare this to the per worker production function.
By dividing both sides of the production function by L, I arrive at the per worker production function:
$$\frac{Y}{L} = \frac{K^{\frac{1}{2}}L^{\frac{1}{2}}}{L} = \frac{K^{\frac{1}{2}}}{L^{\frac{1}{2}}}$$ which is the usual finding of $y = k^{\frac{1}{2}}$.
What is the per-population production function? I divide both sides of the production function by P:
$$\frac{Y}{P} = \frac{K^{\frac{1}{2}}L^{\frac{1}{2}}}{P}$$ or $y = \frac{K^{\frac{1}{2}}L^{\frac{1}{2}}}{P}$. However, since $L = (1 - \psi)P$ it must be that $P = L/(1 - \psi)$.

Substituting this into the per-person production function gives: $y = \dfrac{K^{\frac{1}{2}}L^{\frac{1}{2}}}{L/(1 - \psi)}$. Simplifying the right

hand side of this equation gives the per population production function: $y = (1 - \psi)k^{\frac{1}{2}}$. In other

words, output per person is a function of capital per laborer (k) and $\psi$. One could further write $y$ as : $y = k^{1/2}$ where the italicized $k$ represents the per-population value of capital.

b. Given the evolution of capital through time is given by $K_{t+1} = (1 - \delta)K_t + sY_t$, solve for the per-population equation that describes the evolution of capital over time.

Dividing both sides of this equation by P gives: $\dfrac{K_{t+1}}{P} = (1-\delta)\dfrac{K_t}{P} + s\dfrac{Y_t}{P}$ or $k_{t+1} = (1 - \delta)k_t + sy_t$

(where the italicized variables represent the per-population—as opposed to the per worker—variables). Note: the steady state occurs where $\Delta k_t = 0$ and imposing this on our evolution equation implies the steady state happens when $\delta k_t = sy_t$.)

c. Use the equation found in parts a and b to solve for the steady state level of capital per population and output per population. How does this compare to the steady state level of capital per worker and output per worker?
Since the steady state occurs when $\delta k_t = sy_t$ I substitute the value of $y = k^{1/2}$ into this equation and find $\delta k_t = sk^{1/2}$. Solving for the value of $k$ gives $k = (s/\delta)^2$. Note: solving for the per-population steady state works exactly as the per worker steady state. I plug this value of capital per population into the per-population production function and find $y = s/\delta$.

Since $y = Y/P = Y/[L/(1-\psi)] = (1 - \psi)y$ where $y$ is the per worker amount of output per capita, I can

solve for $y = \dfrac{s}{(1-\psi)\delta}$. Output per worker is higher than output per population but only by the

constant ratio $1/(1-\psi)$.

d. What happens to the steady state level of capital per population as $\psi$ falls to zero? Explain.
Outpute per worker and output per population variables are the same. However, regardless of the value of $\psi$, the relationships are simply constant transformations of each other.


B Level Questions

4. Suppose the economy of Marineland can be described by the following equations:
$$y = k^{\alpha}$$
$$0 < \alpha < 1$$
$$\text{savings} = s\,k^{\alpha}$$
$$\text{depreciation} = (\delta + n)k$$

a. Solve for the steady state level of capital per capita, output per capita, and consumption per capita.
The steady state level of capital occurs when savings per person equals depreciation per person, or:

$$sk^{\alpha} = (\delta + n)k \Rightarrow \frac{s}{(\delta + n)} = k^{1-\alpha} \Rightarrow \left[\frac{s}{(\delta + n)}\right]^{\frac{1}{1-\alpha}} = k^{*}$$

. Given this level of capital per person, output per person is found by inputting this level of k into the production function:

$$y^* = \left(\left[\frac{s}{(\delta+n)}\right]^{\frac{1}{1-\alpha}}\right)^{\alpha} = \left[\frac{s}{(\delta+n)}\right]^{\frac{\alpha}{1-\alpha}}$$

. Since consumption is the fraction of income not saved,

$$c^* = (1-s)\left[\frac{s}{(\delta+n)}\right]^{\frac{\alpha}{1-\alpha}}$$

consumption per person must equal: .

b. Solve for the golden rule level of savings. At this level of savings, what is the steady state level of output per capita, capital per capita, and consumption per capita?

The golden rule level of savings occurs at the savings rate that maximizes consumption. To find this, we first must find the level of capital that gives the maximum of consumption. Consumption is maximized when the slope of the production function is equal to the slope of the depreciation curve, or:

$$\alpha k^{\alpha-1} = (\delta+n) \Rightarrow k^{\alpha-1} = \frac{(\delta+n)}{\alpha} \Rightarrow k_{GR} = \left[\frac{(\delta+n)}{\alpha}\right]^{\frac{1}{\alpha-1}} = \left[\frac{\alpha}{(\delta+n)}\right]^{\frac{1}{1-\alpha}}$$

.

From here, one needs to find the savings rate that yields this level of capital. But, if you look at the steady state level of capital found in part a, you will notice that the steady state level of capital equation looks similar to the amount of capital required to get to the golden rule. As a matter of fact, simple comparison between these equations would indicate that for an economy to reach the golden rule, it should set its level of savings equal to $\alpha$. If $s = \alpha$, then

$$y_{GR} = \left(\left[\frac{\alpha}{(\delta+n)}\right]^{\frac{1}{1-\alpha}}\right)^{\alpha} = \left[\frac{\alpha}{(\delta+n)}\right]^{\frac{\alpha}{1-\alpha}}$$

and $$c_{GR} = (1-\alpha)\left[\frac{\alpha}{(\delta+n)}\right]^{\frac{\alpha}{1-\alpha}}$$ .

A second way of approaching this problem is to realize that the golden rule level of capital (and savings) occurs when the steady state level of consumption is maximized. In part a of this problem,

$$c^* = (1-s)\left[\frac{s}{(\delta+n)}\right]^{\frac{\alpha}{1-\alpha}}$$

we found that the steady state level of consumption is . One simply needs to choose the level of s that maximizes this function which can be done using calculus (simply take the derivative of c* with respect to s, set the derivative equal to zero and solve for s).

c.  Suppose Marineland was saving at the golden rule level of savings and in the steady state.  At time $t_0$, a hurricane strikes Marineland destroying half of the country's capital but not killing any citizens.  Draw the time sequences of the subsequent values of y, c, i=s, and total output.  An example of time sequences is given in figure 4-10 on p. 198 and 199 of Mankiw.
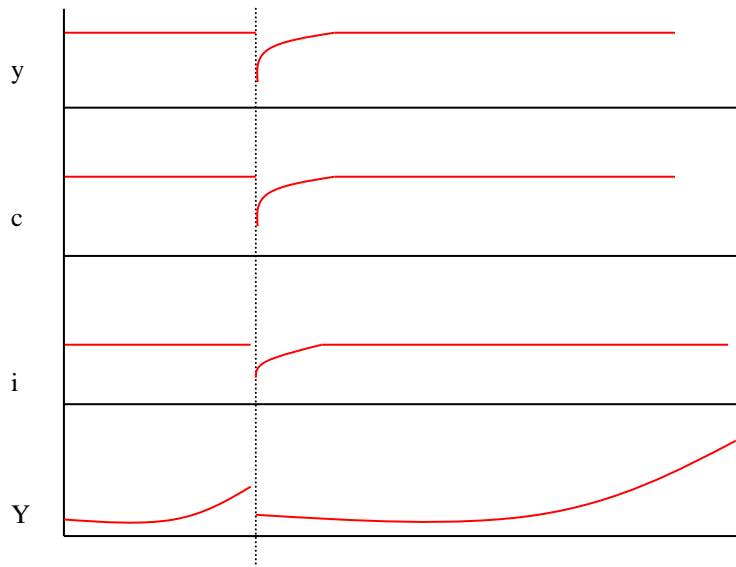


d.  Starting again at the steady state with the golden rule level of savings, suppose at time $t_0$ Marineland had a sudden influx of immigration that doubled the size of the labor force.  After this influx, the amount of labor remains steady (at twice its original level).  Draw the time sequences of the values of y, c, i=s, and total output after time $t_0$.
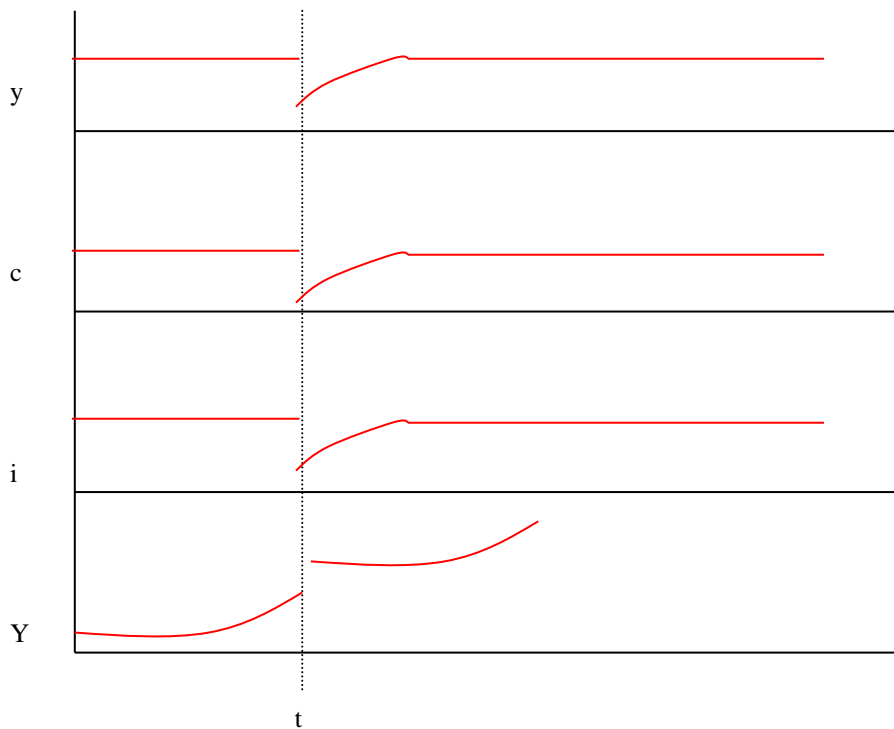
5.      In the Solow Model with population and technological growth, consumption is maximized when the slope of the production function (the MPK) is equal to the slope of the depreciation curve $(n + \delta + g)$ where n measures the population growth rate and g measures the technological growth rate.  A good first guess at American marginal productivity of capital is .12 (a 1 unit increase in capital leads to a .12 unit increase in production).  Good guesses at the American depreication rate is 4%, and since real GDP has grown at about 3% for the last 30 years, a good guess for $n + g = .03$ (see section 8-2 of Mankiw for details).  Given these figures, is America at the Golden Rule level of consumption?  How can you tell?  If not, explain what America needs to do to get closer to the Golden Rule.

<span style="color:red">See pages 211-212 of Mankiw.</span>

Do problems #3 and 6 on p. 206 of Mankiw.
Do problem #1 in problems and applications on p. 227 of Mankiw.

<u>A Level Questions</u>

6.      Imagine the country of Japan could be described as a Solow model of the economy with the equations:

$$y = k^{\frac{1}{4}} \qquad\qquad s = .10 \qquad\qquad \delta = .04$$

a.  Solve for the steady state level of capital and output.  (8)

<span style="color:red">The steady state occurs when savings equals investment.</span>

<span style="color:red">$.1k^{\frac{1}{4}} = .04k \Rightarrow k^{\frac{3}{4}} = 2.5 \Rightarrow k^* = 2.5^{\frac{4}{3}} = 3.393$.</span>

<span style="color:red">$y = k^{\frac{1}{4}} = \left(2.5^{\frac{4}{3}}\right)^{\frac{1}{4}} = 2.5^{\frac{1}{3}} = 1.357$.</span>

<span style="color:red">The steady state level of per capita capital is 3.393 and the steady state level of per capita output is 1.357.</span>

b. Imagine that in 1900, the country of Japan had 3 units of capital per person and followed the Solow growth model found in part a. On the plots below, graph the Solow model's prediction of y, k, and Y between 1900 and 1945.
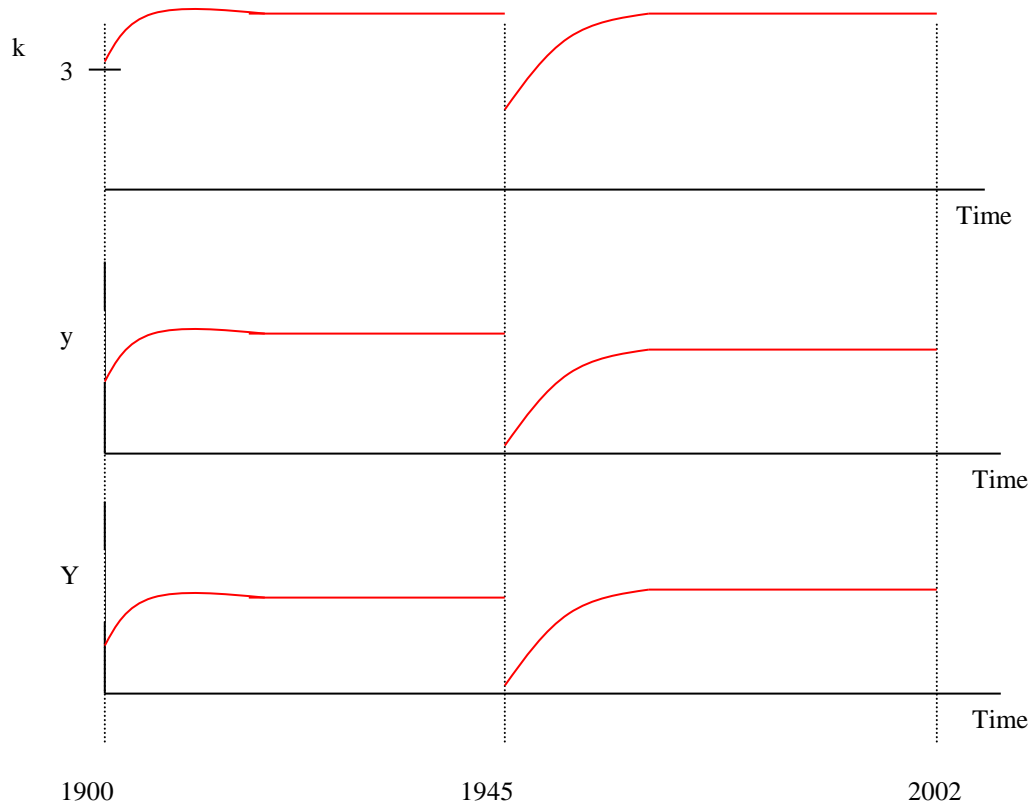


c. In 1945, Japan was the target of a number of large, heavy, explosive devices. Let's imagine that these devices destroyed half of the country's capital without damaging the Japanese population. On the above plot, show the effect of the capital destruction on y, k, and Y between 1945 and 2002. Be sure you show the Japanese economy achieving steady state by 2002.

d. In the plot below, show what happens to the Japanese real interest rate over the time period described in parts b and c. Hint: Imagine that investment demand grows at the same rate as real GDP.
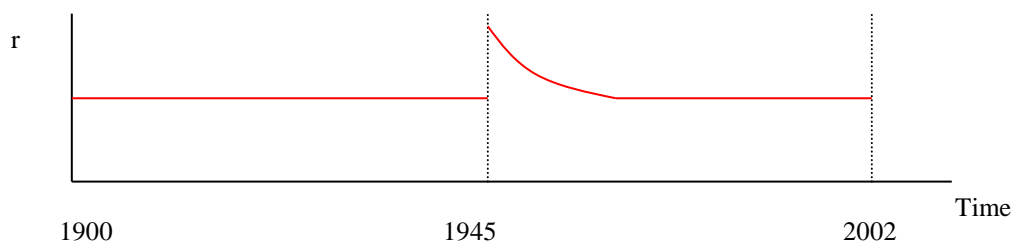
e. The savings rate of 10% in the Japanese economy is not the golden rule savings rate. Solve for the golden rule savings rate for the Japanese economy. What is the significance of the golden rule?

<span style="color:red">The golden rule level of savings is the percentage of output devoted to investment that maximizes consumption. This maximum occurs when the slope of the production function equals the slope of the depreciation curve or:</span>

<span style="color:red">$\frac{1}{4}k^{-3/4} = .04 \Rightarrow k_{GR} = .16^{-4/3}$. This is the level of capital that achieves the golden rule. To find the savings rate that yields this capital, one needs to remember that in the steady state, savings equals depreciation. Plugging this golden rule level of capital into the steady state equation gives:</span>

<span style="color:red">$s\left(.16^{-4/3}\right)^{1/4} = .04 \times .16^{-4/3} \Rightarrow s = \frac{.04 \times .16^{-4/3}}{.16^{-1/3}} = \frac{1}{4}$. A country that saves 25% of its income achieves the golden rule level of capital.</span>

f. Below is a graph of per capita annual growth rates in Japan. Do they follow what the Solow model predicts? Why or why not?

**% Change in Annual Per Capita Real GDP**
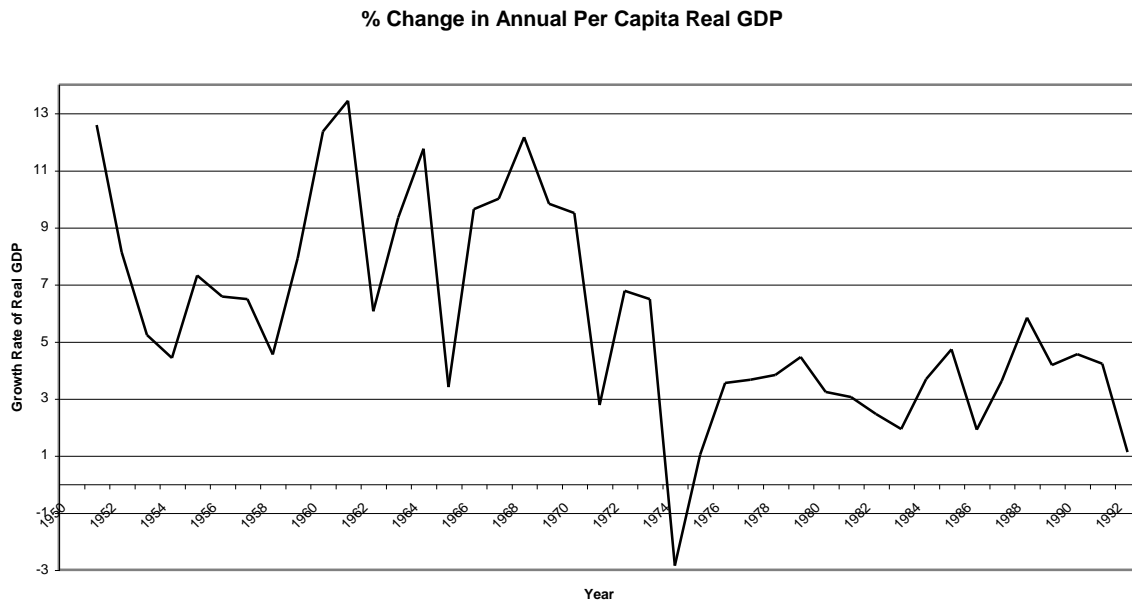


<span style="color:red">This chart seems to indicate that when Japan was economically devastated after WW2, it grew very fast and later, after it got larger, its average growth rate fell. This is what the Solow model implies; due to diminishing returns to capital, a country with little capital will grow fast as it adds capital but a country while a country with much capital will grow more slowely.</span>

7.     Consider the following Solow growth model:
$$y = k^{.25} \qquad s = .08 \qquad n = .02 \qquad \delta = .08$$
a.  Solve for the steady state level of capital per capita and output per capita.

The steady state level of capital is given by:  $.08k^{.25} = .1k \Rightarrow k^{.75} = \dfrac{.08}{.1} \Rightarrow .7426$

At this level of capital, the steady state level of output is given by:  $(.7426)^{.25} = .928$

b.  Is this economy at the golden rule?  If not, give one governmental policy that will move this economy closer to the golden rule.
This economy is not at the steady state.  The steady state level of savings occurs when s = .25.  In order to move close to the golden rule, this policies should be imposed to increase the level of savings (tax breaks for savings, higher sales tax, etc.)

c.  Imagine the economy described in part a has 1 unit of capital per person.  On the graphs below, chart the progress over time of this economy.  Assume that by time $t_1$, this economy achieves the steady state.  Be sure to plot what happens to this economy over time after it reaches the steady state.
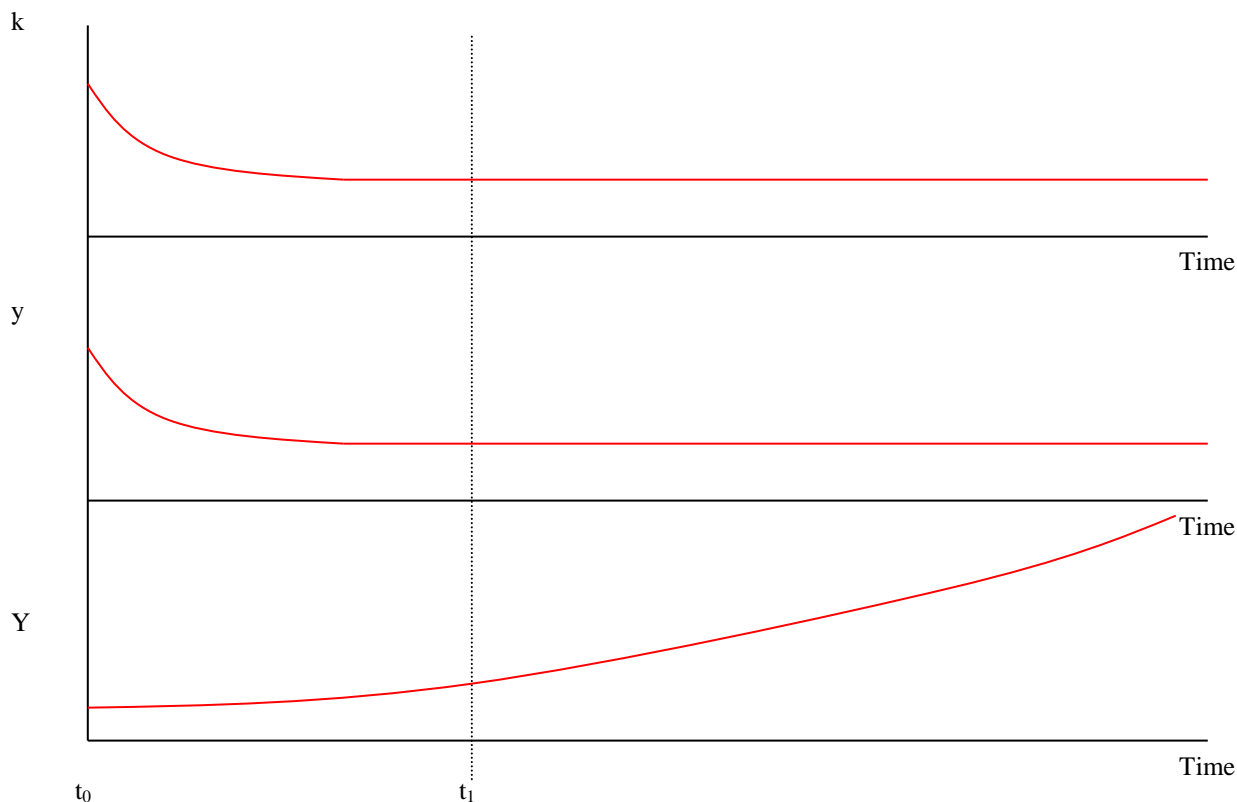
d. Imagine that at time $t_2$ the depreciation rate rose from 8% to 10%. On the charts below demonstrate the impact of this change on the Solow economy. Be sure to show what was happening in the economy before time $t_2$ and after the economy reaches its new steady state.