MEC-E1005

MODELLING IN APPLIED MECHANICS 2024

WEEK 17: MATHEMATICA WORKSHOP

Tue 12:15-14:00 (JF)

LEAST-SQUARES METHOD

In a typical design, dataset of an experiment $\{\ldots,(x_i, f_i),(x_{i+1}, f_{i+1}),\ldots\}$ is considered as sampling of the underlying continuous *dependent quantity* $f(x)$ at values $\{\ldots, x_i, x_{i+1},\ldots\}$ of the *independent quantity* x. In further processing of data, one may

- ☐ use the dataset to find a continuous approximation $g(x)$ to $f(x)$. Thereafter finding the value at any point, calculation of derivatives, integration etc. with *generic methods* is possible.

- ☐ use the dataset directly to find, e.g., derivatives at the sampling points, integrals, etc. using *dedicated methods* like difference approximations and quadratures (numerical integration).

Although the details of the methods differ, the results at the sampling points may not differ too much from the engineering viewpoint.

LEAST-SQUARES APPROXIMATION

Finding an approximation $g(x)$ to function $f(x)$ is one the basic tasks in numerical mathematics. In the Least-Squares-Method, approximation $g(x)=\sum a_i N_i(x)=\mathbf{N}^{\mathrm{T}}\mathbf{a}$ follows from steps ($N_i(x)=x^i$ is the usual choice)

Error measure: $\Pi(\mathbf{a})=\frac{1}{2}\int_0^L (g-f)^2 dx=\frac{1}{2}\int_0^L (\mathbf{N}^{\mathrm{T}}\mathbf{a}-f)^2 dx$,

Minimizer: $\mathbf{Ka}-\mathbf{F}=\mathbf{0}$ where $\mathbf{K}=\int_0^L \mathbf{N}\mathbf{N}^{\mathrm{T}}dx$ and $\mathbf{F}=\int_0^L \mathbf{N}f dx$

Multipliers: $\mathbf{a}=\mathbf{K}^{-1}\mathbf{F}$.

In practice, multipliers are often solved from linear equation system $\mathbf{Ka}=\mathbf{F}$. The method works in the same manner irrespective of the series approximation.

LEAST-SQUARES FIT

Finding a fit $g(x)$ to $f(x)$ known at discrete points $f_i = f(x_i)$ only is another version of finding an approximation. In the Least-Squares-Method, the fit $g(x) = \sum_j a_j N^j(x) = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ follows from steps ($N^j(x) = x^j$ is the usual choice)

Error measure $\Pi(\mathbf{a}) = \frac{1}{2}\sum_i (f_i - \mathbf{N}_i^{\mathrm{T}}\mathbf{a})^2$

Minimizer: $\mathbf{Ka} - \mathbf{F} = \mathbf{0}$ where $\mathbf{K} = \sum_i \mathbf{N}_i \mathbf{N}_i^{\mathrm{T}}$ and $\mathbf{F} = \sum_i \mathbf{N}_i f$,

Multipliers: $\mathbf{a} = \mathbf{K}^{-1}\mathbf{F}$.

In practice, multipliers are often solved from linear equation system $\mathbf{Ka} = \mathbf{F}$. The method works in the same manner irrespective of the series approximation.

FOURIER TRANSFORM

The Fourier series (various forms exist) can be used to represent a function as the sum of harmonic terms. For example, the sine-transformation pair for a function $a(x)$ $x \in [0, L]$ with vanishing values at the end points is given by

$$\alpha_j = \frac{2}{L}\int_0^L \sin(j\pi\frac{x}{L})a(x)dx \quad j \in \{1, 2, \ldots\} \quad \Leftrightarrow \quad a(x) = \sum_{j\in\{1,2,\ldots\}} \alpha_j \sin(j\pi\frac{x}{L}).$$

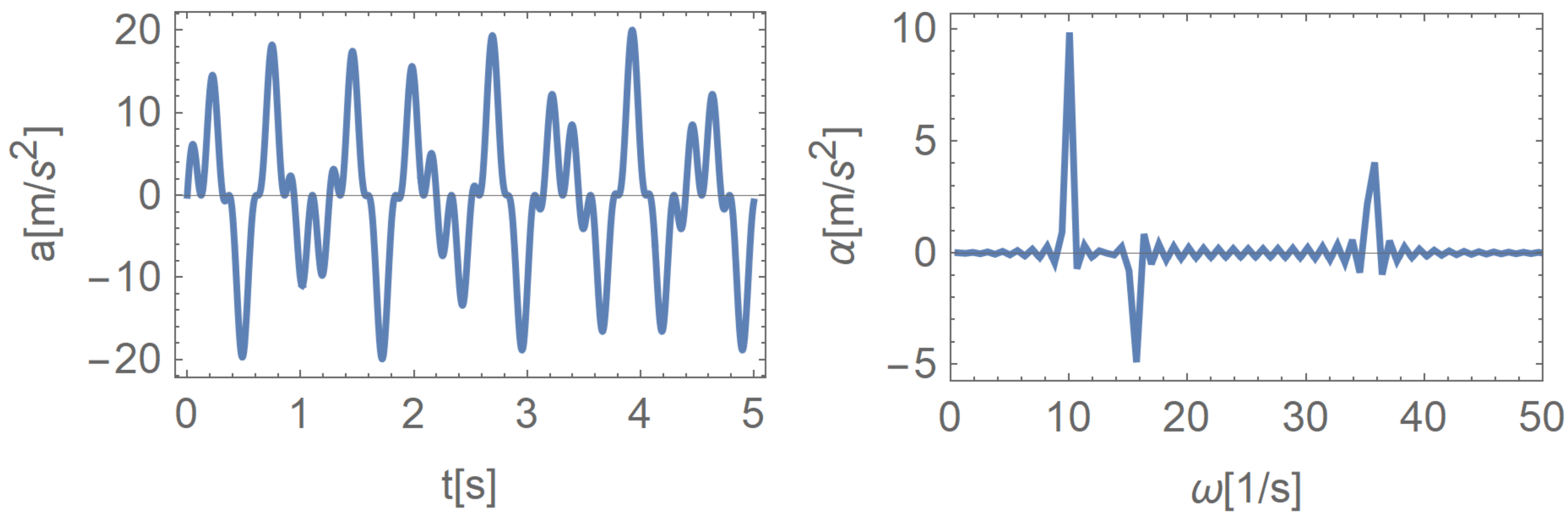
The transformation pair is based on the orthogonality of the modes

$$\int_0^L \sin(j\pi\frac{x}{L})\sin(l\pi\frac{x}{L})dx = \frac{L}{2}\delta_{jl} \quad \text{(Kronecker delta).}$$

Transformation (with respect to time) can be used to analyze frequency contents of data, filtering, to find the combination of the terms of the generic series solution for bar and string models satisfying the initial conditions, etc.

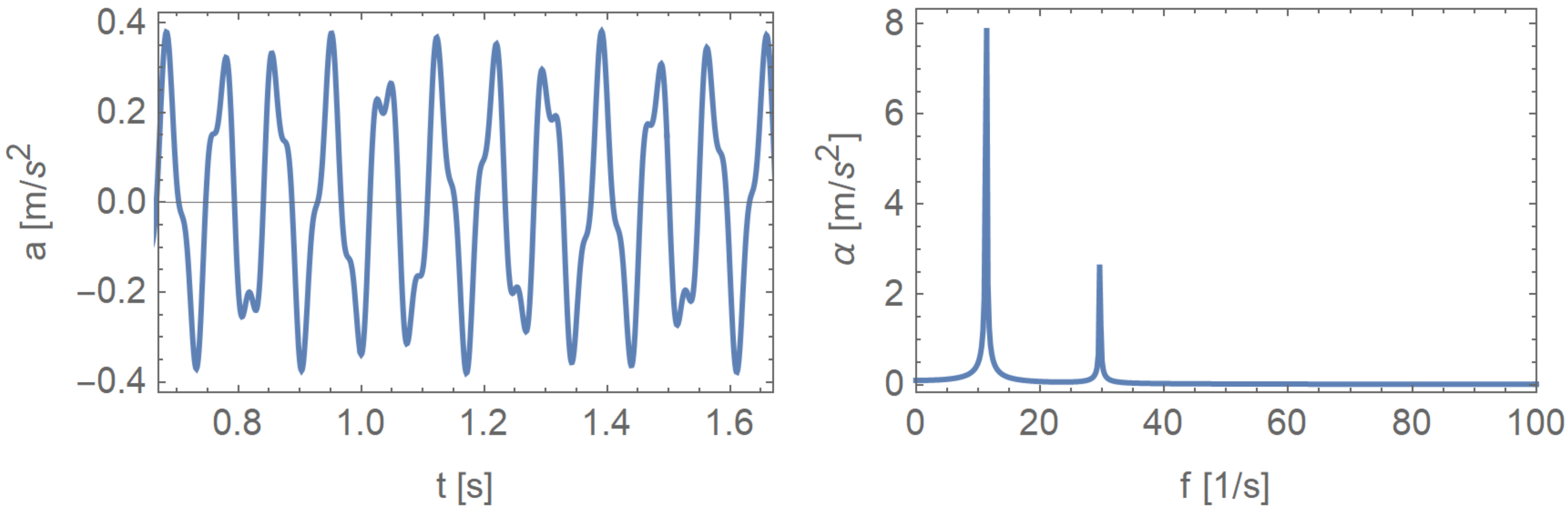
FREQUENCY CONTENTS OF DATA

Fourier transform (with respect to time) can be used, e.g., to analyze frequency contents of data, filtering of data etc. As an example, transform (right) of the measured acceleration (left), imply $a(t) = 10\sin(10t) - 5\sin(15.6t) + 5\sin(35.6t)$ (in appropriate units)



In filtering, one may just omit, e.g., components having frequencies over some value or maybe components of amplitudes of small values depending on the application.

VIBRATION EXPERIMENT

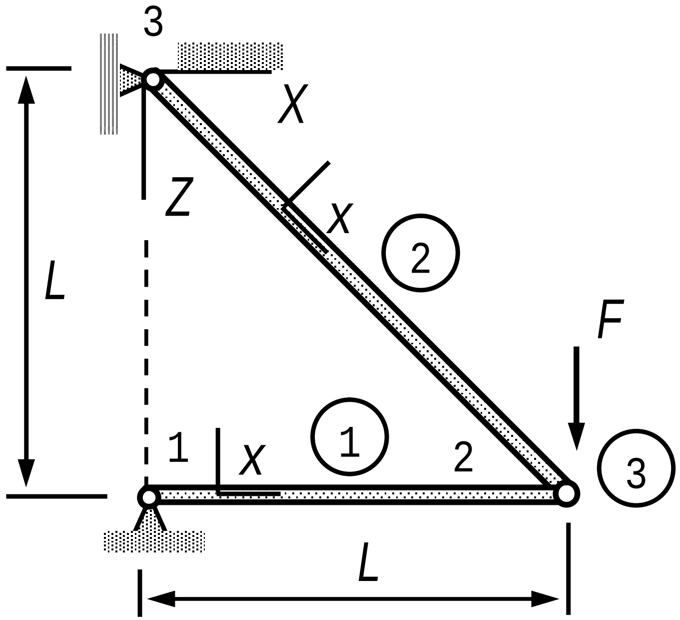


Experimental data consists of the acceleration time-series measured by the accelerometer at one point. In processing of data, the time-acceleration representation is transformed to frequency-mode magnitude form by Discrete Fourier Transform (DFT).

FE-CODE OF MEC-E1050/ MEC-E8001

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	$\{\{E\},\{2\sqrt{2}A\}\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}



STRUCTURE

"Structure is a collection of *elements* (earlier structural parts) connected by *nodes* (earlier connection points). Displacement of the structure is defined by nodal translations and rotations of which some are known and some unknown."

$prb = \{ele, fun\}$ where

$ele = \{prt_1, prt_2, \ldots\}$ .... elements

$fun = \{val_1, val_2, \ldots\}$ .... nodes

Elements

$prt = \{typ, pro, geo\}$ where

$typ = \mathrm{BAR} \mid \mathrm{TORSION} \mid \mathrm{BEAM} \mid \mathrm{RIGID} \mid \ldots \mid$ .... model

$pro = \{p_1, p_2, \ldots, p_n\}$ ....properties

$$geo = \text{Point}[\{n_1\}] \,|\, \text{Line}[\{n_1, n_2\}] \,|\, \text{Triangle}[\{n_1, n_2, n_3\}] \,|\, \ldots \,|$$ .... geometry

Nodes

$val = \{crd, tra, rot\}$ where

$crd = \{X, Y, Z\}$ .... structural coordinates

$tra = \{u_X, u_Y, u_Z\}$ .... translation components

$rot = \{\theta_X, \theta_Y, \theta_Z\}$ ....rotation components

ELEMENTS

Elements represent the structural parts modelled as solids, plates, beams, or rigid bodies or their simplified versions, external point and boundary forces and moments.

Constraint

$\{\text{JOINT},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\}\},\text{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\text{JOINT},\{\},\text{Line}[\{n_1,n_2\}]\}$ ....displacement constraint

$\{\text{RIGID},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\},\{\underline{\theta}_X,\underline{\theta}_Y,\underline{\theta}_Z\}\},\text{Point}[\{n_1\}]\}$ ... displacement/rotation constraint

$\{\text{RIGID},\{\},\text{Line}[\{n_1,n_2\}]\}$ .... rigid constraint

$\{\text{SLIDER},\{n_X,n_Y,n_Z\},\text{Point}[\{n_1\}]\}$ ....slider constraint

Force

$\{\text{FORCE},\{F_X,F_Y,F_Z\},\text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X,F_Y,F_Z,M_X,M_Y,M_Z\},\text{Point}[\{n_1\}]\}$ .... point load

$\{\mathrm{FORCE},\{f_X, f_Y, f_Z\},\mathrm{Line}[\{n_1,n_2\}]\}$ .... distributed force

$\{\mathrm{FORCE},\{f_X, f_Y, f_Z\},\mathrm{Polygon}[\{n_1,n_2,n_3\}]\}$ .... distributed force

Beam model

$\{\mathrm{BAR},\{\{E\},\{A\},\{f_X, f_Y, f_Z\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....bar mode

$\{\mathrm{TORSION},\{\{G\},\{J\},\{\{m_X,m_Y,m_Z\}\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ .... torsion mode

$\{\mathrm{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz}\},\{f_X, f_Y, f_Z\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....beam

$\{\mathrm{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz},\{j_X, j_Y, j_Z\}\},\{f_X, f_Y, f_Z\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....beam

Plate model

$\{\mathrm{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\mathrm{Polygon}[\{n_1,n_2,n_3\}]\}$........................thin slab mode

$\{\mathrm{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\mathrm{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$...................thin slab mode

$\{\mathrm{PLATE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\mathrm{Polygon}[\{n_1,n_2,n_3\}]\}$.......................................plate

Solid model

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\text{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z,m_X,m_Y,m_Z,\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

OPERATIONS

Operations act on structure as defined by prb. The main operations are solving the unknowns in displacement analysis and displaying the problem definition in a formatted form.

$prb = \mathrm{REFINE}[prb]$ ....refine structure representation

$\mathrm{Out} = \mathrm{FORMATTED}[prb]$ .... display problem definition

$\mathrm{Out} = \mathrm{STANDARDFORM}[prb]$ .... display virtual work expression

$sol = \mathrm{SOLVE}[prb]$...solve the unknowns