# $\boxed{\text{percolation}}$ lecture notes

kieran ryan

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# 1. definitions & measures

## 1.1 graphs

**def** a graph $G = (V, E)$ is a pair $V, E$, where $V$ is a set (the vertices), $E$ a set of sets $\{x, y\}$, $x, y \in V$ of pairs of elements of $V$ (the edges).

**rmk** we allow multiple edges between vertices, but we won't see any in this course

**rmk** think of pictures:


**def** we say $G$ is finite if $V$ and $E$ are finite sets, and infinite otherwise.

**defs** • $\mathbb{Z}^d = (V, E)$ the hypercubic lattice dimension $d$, where

- $V = \{(x_1, \dots, x_d) : x_i \in \mathbb{Z}\} = \mathbb{Z}^d$
- $E = \{\{x, y\} \subseteq \mathbb{Z}^d : \|x-y\|_1 = 1\}$

$\|x\|_1 = \sum_{i=1}^{d} |x_i|$.

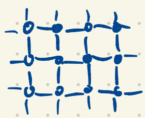
• we often write just $\mathbb{Z}^d$ for the vertex set.

• for $\{x,y\} \in E$, we often just write $xy$, & say $x \sim y$, and that $x$ and $y$ are neighbors or adjacent.

$d=1$

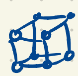
line

$d=2$



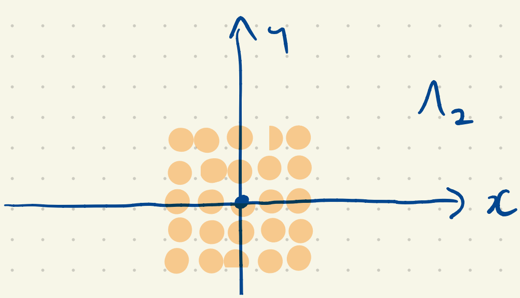
square lattice

$d=3$



cubic lattice

**def** $\Lambda_n := \{-n, \dots, n\}^d = B_{\|\cdot\|_1}(0, n)$ is the "box of size n around 0"



**def** let $S \subset \mathbb{Z}^d$ (set of vertices). the (vertex) boundary of $S$ is

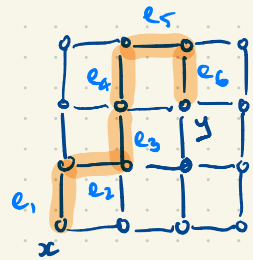
$$\partial S = \{ x \in S : \exists y \in \mathbb{Z}^d \setminus S , x \sim y \}$$

**eg** $\partial \Lambda_n = \{ x \in \mathbb{Z}^d : \|x\|_\infty = n \}$

where $\|x\|_\infty = \max\{ x_i : 1 \leq i \leq d \}$.

**def** a self-avoiding path $\gamma$ from $x \in V$ to $y \in V$ is a sequence $\gamma = (e_1, \dots, e_L)$ of <u>distinct edges</u> st. $x \in e_1$, $y \in e_L$,

$$\forall i \neq j ; \quad |e_i \cap e_j| = \begin{cases} 1 & \text{if } j = i \pm 1 \\ 0 & \text{o/w} \end{cases}$$



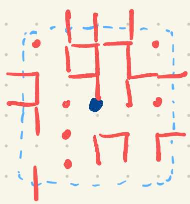
## 1.2 bernoulli percolation on a finite graph

- we want to keep each edge with prob $p$, delete with prob $1-p$.

**def** a (bond) percolation configuration is a function $\omega : E \to \{0,1\}$, or $\omega = (\omega_e)_{e\in E} \in \{0,1\}^E =: \Omega$.

**rmk** • $\{0,1\}^E \overset{\text{bijection}}{\longleftrightarrow} 2^E$ (subsets of $E$)

$\omega \longleftrightarrow E_\omega := \{e : \omega_e = 1\}$



**def** let $\omega \in \{0,1\}^E$.

- $e \in E$ is called open if $\omega_e = 1$
  closed if $\omega_e = 0$

- a cluster is a connected component of $G_\omega$ (set of vertices & edges).

**rmk** cluster can be just 1 vertex

**def** on a finite graph $G = (V, E)$:

- $\Omega := \{0,1\}^E$
- let $p \in [0,1]$.

  $\mathbb{P}_p[\omega] := p^{o(\omega)}(1-p)^{c(\omega)}$ $\quad \forall \omega \in \Omega$

  $o(\omega) = |E_\omega|$, $c(\omega) = |E| - |E_\omega|$.

- for any $A \subset \Omega$, we define

  $$\mathbb{P}_p[A] := \sum_{\omega \in A} \mathbb{P}_p[\omega].$$

- $\mathbb{F} := 2^\Omega$ is called the set of events

**eg** $G = \Lambda_n$, $A = \{x \leftrightarrow y \text{ in } \Lambda_n\}$

ie $\exists$ open path $\gamma : x \to y$, $\gamma = (e_1, \ldots, e_L)$, $e_i \in \Lambda_n$.

**def** our function $\mathbb{P}_p$ is a probability measure on $\Omega$

let $\Omega$ any finite set. $\mathbb{P}_p$ is a measure on $\Omega$ if:

- $\mathbb{P}_p : 2^\Omega \to \mathbb{R}$ $\quad \boxed{*}$
- $\mathbb{P}_p[A] \geq 0$ $\quad \forall A \in 2^\Omega$
- $\mathbb{P}_p[\emptyset] = 0$
- $\mathbb{P}_p\left[\bigcup_{i=1}^\infty A_i\right] = \sum_{i=1}^\infty \mathbb{P}_p[A_i]$ $\quad \forall A_i$ pairwise disjoint.

  (countable additivity)

## 1.3 bernoulli percolation on infinite graphs: we need $\sigma$-algebras

- we want to do the above when $G = \mathbb{Z}^d$. we hit a problem: we cannot define $\mathbb{P}_p$ on $2^\Omega$, $\Omega = \{0,1\}^{E(\mathbb{Z}^d)}$ consistently.
- we need to use a subset $\mathbb{F} = 2^\Omega$ to be our set of events. this $\mathbb{F}$ should have certain properties:

**def** let $\Omega$ be a set. $\mathbb{F}$ a nonempty set of subsets of $\Omega$ is a $\sigma$-algebra if
- $\Omega \in \mathbb{F}$

and $\mathbb{F}$ is closed under:
- complement $A \in \mathbb{F} \Rightarrow \Omega \setminus A \in \mathbb{F}$
- countable unions $A_i \in \mathbb{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathbb{F}$

**ex** $\emptyset, \Omega \in \Sigma$, closed also under countable intersections

**def** $\mathbb{P} : \mathbb{F} \to \mathbb{R} \cup \{\infty\}$ is a measure if it satisfies $\boxed{*}$ with $2^\Omega$ replaced with $\mathbb{F}$.

rmk sets in $\Omega$ are called measurable, they are those sets $\mathbb{P}$ can assign a size to.

rmk $(\Omega, \mathbb{F}, \mathbb{P})$ called a measure space

def $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space if it is a measure space, and

- $\mathbb{P} : \mathbb{F} \to [0,1]$,
- $\mathbb{P}[\Omega] = 1$.

& we call $\mathbb{P}$ a probability measure

eg for any $\Omega$ countable, and $\mathbb{F} = 2^{\Omega}$, $\mathbb{P}$ probability measure, $(\Omega, \mathbb{F}, \mathbb{P})$ is a discrete probability space. the $\sigma$-algebra is very concrete and easy to deal with

rmk more generally, we work with $\sigma$-algebras generated by a concrete family of subsets.

def an event $A \subset \{0,1\}^E$ is a cylinder event if it only depends on finitely many edges

ie $A = A_1 \times \{0,1\}^{E \setminus E_1}$, $A_1 \subset \{0,1\}^{E_1}$, $E_1$ finite.

**eg** $\{x \leftrightarrow y \text{ in } \Lambda_n\}$

ie $\exists$ path $\gamma : x \to y$, $\gamma = (e_1, \dots, e_k)$, $e_i \in \Lambda_L$.
open

- we certainly want cylinder events in our $\sigma$-algebra $\mathbb{F}$.

rmk: defining a measure on $\mathbb{R}$, one wants

$$\mathbb{P}\big[(a,b)\big] = b-a \quad , \quad \text{so}$$

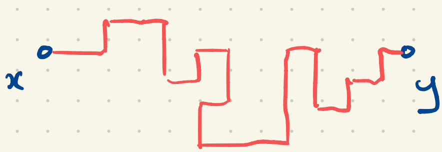
want finite intervals to be contained in $\mathbb{F}$.

**def** if $\mathbb{A} \subset 2^{\Omega}$ is a set of subsets of $\Omega$, then $\sigma(\mathbb{A}) := \{E \subset \Omega : \forall\ \sigma\text{-algs } \mathbb{F} \subset 2^{\Omega} \text{ containing } \mathbb{A}, E \in \mathbb{F}\}$; we call this the $\sigma$-alg generated by $\mathbb{A}$.

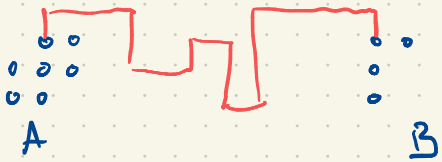
**def** on $\Omega = \{0,1\}^{E(\mathbb{Z}^d)}$, we let $\mathbb{F} = \sigma(\mathbb{A})$, $\mathbb{A} = \{\text{cylinder sets in } \mathbb{Z}^d\}$.

**eg** some events in $\mathbb{F}$. let $x, y \in \mathbb{Z}^d$, $A, B \subset \mathbb{Z}^d$.
(below)

**def** • $\{x \leftrightarrow y\} := \{\omega : \exists \text{ open path from } x \text{ to } y\}$



- $\{x \not\leftrightarrow y\} = \{x \leftrightarrow y\}^c$
- $\{A \leftrightarrow B\} = \{\exists\, x \in A,\ y \in B \ : \ x \leftrightarrow y\}$



- $\{x \leftrightarrow \infty\} = \{x$ belongs to an $\infty$ cluster$\}$
- $\{$percolation$\} = \{\exists$ an $\infty$ cluster$\}$

ex these sets are measurable (ie. in $\mathbb{F}$).

rmk when defining measure on $\mathbb{R}$, we set $\mathbb{A} = \{$finite unions of open/closed/clopen intervals$\}$, & set $\mathbb{F} = \sigma(\mathbb{R})$.

- we have $\mathbb{P}_p$ defined on $\mathbb{A}$, we've defined $\mathbb{F} = \sigma(\mathbb{A})$. it remains to extend $\mathbb{P}_p$ & define it on $\mathbb{F}$.

**thm** (carathéodory's extension thm)

let $\mathbb{A} \subset 2^\Omega$ be a set of subsets of $\Omega$, st.

- $\emptyset \in \mathbb{A}$
- $A, B \in \mathbb{A} \Rightarrow B \setminus A \in \mathbb{A}, \quad A \cup B \in \mathbb{A}$

(we call $\mathbb{A}$ a ring). let $\mathbb{P} : \mathbb{A} \to [0, \infty]$ st.

- $\mathbb{P}(\emptyset) = 0$
- $A_i \in \mathbb{A} \ \forall i \in \mathbb{N}$ disjoint with $\dot\bigcup_{i=1} A_i \in \mathbb{A}$

$$\Rightarrow \mathbb{P}\left[\dot\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$$

(we call $\mathbb{P}$ a pre-measure)

**then** $\exists$ extension of $\mathbb{P}$ to a measure on $\sigma(A)$.

moreover, if $\exists \ X_i \in \mathbb{A}, (i \in \mathbb{N})$, disjoint, st.

- $\mathbb{P}[X_i]$ finite
- $\Omega = \dot\bigcup_{i=1}^{\infty} X_i$

(we call $\mathbb{P}$ $\sigma$-finite) **then** $\mathbb{P}$'s extension is unique

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**lem** (union bound). let $(\Omega, \mathbb{F}, \mathbb{P})$ be a measure space. for all $A_i \in \mathbb{F}$,

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \le \sum_{i=1}^{\infty} \mathbb{P}[A_i]$$

**cor** let $\mathbb{P}$ on $(\Omega, \mathbb{F} = \sigma(\mathbb{A}))$ be constructed as above. then $\forall$ events $A \in \mathbb{F}$, $\forall \varepsilon > 0$, $\exists$ event $A_\varepsilon \in \mathbb{A}$ such that

$$\mathbb{P}[A \triangle A_\varepsilon] \leq \varepsilon$$

**rmk** when $\Omega = \{0,1\}^E$, we call this approximation by cylinder sets.


**rmk** the above is an example of a product measure. let $\{(\Omega_i, \mathbb{F}_i, \mu_i)\}_{i \in \mathbb{N}}$ be probability spaces.

let $\Omega = \prod_{i=1}^{\infty} \Omega_i$, let $\mathbb{F} = \sigma(\text{cylinder sets})$.

then $\mu$ on cylinder sets as a product measure is a pre-measure, $\exists!$ extension to a measure on $(\Omega, \mathbb{F})$. we often write $\mu = \bigotimes_{i=1}^{\infty} \mu_i$.