10. correlation length

- we return to the sharpness theorem, now armed with the FKG inequality.
- we know that for $p < p_c$, $p^n \le \theta_n(p) \le e^{-cn}$ for some $c > 0$.
- can we say more about the asymptotics of $\theta_n(p)$?

def let $e_1 = (1, 0, \dots, 0)$, $p \in [0,1]$. let

$$\xi(p) = \lim_{n\to\infty} \left[-\frac{1}{n} \log \mathbb{P}_p\left[0 \leftrightarrow ne_1\right]\right]^{-1} ;$$

$\xi(p)$ is known as the correlation length.

thm $\xi(p)$ exists, and for $p < p_c$, $\xi(p)$ is finite.

lem (fekete's lemma) let $(a_n)_{n\ge 0}$ be a sequence of numbers in $[-\infty, \infty)$ st. $a_{n+m} \le a_n + a_m$. (subadditivity). then

$$\lim_{n\to\infty} \frac{a_n}{n} \text{ exists, lies in } [-\infty,\infty) \text{, and equals } \inf\left\{\frac{a_n}{n} : n \ge 1\right\}.$$

<u>proof of thm</u> : let $a_n = -\log \mathbb{P}_p[0 \leftrightarrow ne_1]$.

- if a_n is subadditive, then $\mathbb{P}_p[0 \leftrightarrow ne_1] \leq e^{-cn} \Rightarrow a_n \geq c$,

  we have $\lim_{n\to\infty} \frac{a_n}{n}$ exists & lies in $[c, \infty)$.

  hence $\xi(p)$ exists & lies in $(0, c^{-1}]$.

- $$\mathbb{P}_p[0 \leftrightarrow (n+m)e_1] \geq \mathbb{P}_p[0 \leftrightarrow ne_1,\ ne_1 \leftrightarrow (n+m)e_1]$$

  FKG
  $$\geq \mathbb{P}_p[0 \leftrightarrow ne_1] \cdot \mathbb{P}_p[ne_1 \leftrightarrow (n+m)e_1]$$
  $$= \mathbb{P}_p[0 \leftrightarrow ne_1] \cdot \mathbb{P}_p[0 \leftrightarrow me_1]$$
  translation invariance.

  which gives a_n subadditive. ∎

<u>prop</u> let $p < p_c$. $\exists c > 0$ st. $\forall n \geq 1$,

$$\frac{1}{cn^{d-1}} e^{-\frac{1}{\xi(p)} n} \leq \theta_n(p) \leq cn^{d-1} e^{-\frac{1}{\xi(p)} n}$$

<u>proof</u> : fix $p < p_c$.

upper bound :

- we have $\forall n \geq 1$, $\quad \frac{a_n}{n} \geq \inf\left\{ \frac{a_n}{n} : n \geq 1 \right\} = \frac{1}{\xi(p)}$

$\Rightarrow \forall n \geq 1$, $\quad \mathbb{P}_p[0 \leftrightarrow n e_1] \leq e^{-\frac{1}{\xi(p)} n}$.

- by symmetry, we can choose $x \in \partial \Lambda_n$ with $x_1 = n$ st.

$$\mathbb{P}_p[0 \leftrightarrow x] = \max_{y \in \partial \Lambda_n} \mathbb{P}_p[0 \leftrightarrow y]$$

- by invariance under reflection through $\{n\} \times \mathbb{Z}^{d-1}$, we have
$$\mathbb{P}_p[0 \leftrightarrow x] = \mathbb{P}_p[x \leftrightarrow 2n e_1]$$

$$\overset{\text{FKG}}{\Rightarrow} \quad \mathbb{P}_p[0 \leftrightarrow 2n e_1] \geq \mathbb{P}_p[0 \leftrightarrow x]^2$$

- so we have

$$\theta_n(p) = \mathbb{P}_p[0 \leftrightarrow \partial \Lambda_n]$$
$$\leq \sum_{y \in \partial \Lambda_n} \mathbb{P}_p[0 \leftrightarrow y]$$
$$\leq |\partial \Lambda_n| \cdot \mathbb{P}_p[0 \leftrightarrow x]$$

$$\leq |\partial\Lambda_n| \cdot \mathbb{P}_p[0 \leftrightarrow 2ne_1]^{1/2}$$

$$\leq c \cdot n^{d-1} e^{-\frac{1}{\xi(p)} n}.$$

—

<u>lower bound</u>: for $1 \leq m \leq n$, we have

$$\theta_{n+m}(p) \leq \mathbb{P}_p\left[0 \leftrightarrow \partial\Lambda_m,\ \partial\Lambda_m \xleftrightarrow{\Lambda_{n+m}\setminus\Lambda_m} \partial\Lambda_{n+m}\right]$$

$$\underset{\text{independence}}{=} \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_m] \cdot \mathbb{P}_p\left[\partial\Lambda_m \xleftrightarrow{\Lambda_{n+m}\setminus\Lambda_m} \partial\Lambda_{n+m}\right]$$

$$\leq \theta_m(p) \cdot \mathbb{P}_p[\partial\Lambda_m \leftrightarrow \partial\Lambda_{n+m}]$$

$$\leq \theta_m(p) \sum_{x \in \partial\Lambda_m} \mathbb{P}[x \leftrightarrow \partial\Lambda_{n+m}]$$

$$\leq \theta_m(p) \cdot \theta_n(p) \cdot 8^{d-1} m^{d-1}$$

as $\Lambda_n(x) \subset \partial\Lambda_{n+m}$ $\forall x \in \partial\Lambda_m$.

- let $C = 16^{d-1}$ we have

$$C \cdot (n+m)^{d-1} \theta_{n+m}(p) \leq C \cdot (2n)^{d-1} 8^{d-1} m^{d-1} \theta_m(p)\, \theta_n(p)$$

$$\leq \left(C m^{d-1} \theta_m(p)\right)\left(C n^{d-1} \theta_n(p)\right)$$

hence $b_n := \log\left(C n^{d-1} \theta_n(p) \right)$ is subadditive.

- by fekete,

$$\frac{\log\left(C k^{d-1} \theta_k(p)\right)}{k} = \frac{b_k}{k} \geq \inf\left\{ \frac{b_n}{n} , n \geq 1 \right\} = \lim_{n\to\infty} \frac{b_n}{n}$$

and

$$\lim_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} \frac{\log C n^{d-1}}{n} + \frac{\log \theta_n(p)}{n}$$

$$\geq \lim_{n\to\infty} \frac{\log \mathbb{P}_p[0 \leftrightarrow n e_1]}{n} = -\frac{1}{\xi(p)},$$

hence

$$\theta_k(p) \geq \frac{1}{C k^{d-1}} e^{-\frac{1}{\xi(p)} k}$$

$\blacksquare$

rmk intuitively, $\xi(p)$ is a scale at which things "look like you're at p_c".

prop $\xi : [0,1] \to [0,\infty]$ is continuous, nondecreasing, and satisfies $\xi(0) = 0$, $\xi(p_c) = +\infty$

<u>proof</u> above we proved

- $$\frac{1}{\xi(p)} = \sup_{n \geq 1} \frac{-\log\left(C n^{d-1} \theta_n(p)\right)}{n}$$

$$\Rightarrow \liminf_{p \to p_0} \frac{1}{\xi(p)} \geq \frac{1}{\xi(p_0)} \quad \left(\frac{1}{\xi} \text{ is lower semicontinuous}\right)$$

indeed, $\liminf_{p \to p_0} \sup_{n \geq 1} \underbrace{\frac{-\log\left(C n^{d-1} \theta_n(p)\right)}{n}}_{\alpha_n(p)}$

$$\lim_{p \to p_0} \inf_{p' \geq p} \sup_{n \geq 1} \alpha_n(p) \geq \lim_{p \to p_0} \inf_{p' \geq p} \alpha_n(p) = \alpha_n(p_0)$$

$$\Rightarrow \text{LHS} \geq \sup_{n \geq 1} \alpha_n(p_0) .$$

- similarly, $\frac{1}{\xi(p)} = \inf_{n \geq 1} \frac{-\log\left(\frac{1}{C n^{d-1}} \theta_n(p)\right)}{n}$

$\Rightarrow \frac{1}{\xi}$ upper semicontinuous

- together $\Rightarrow \frac{1}{\xi}$ continuous.
- further, from sharpness it is clear that $\frac{1}{\xi(p)} \begin{cases} > 0 & \text{for } p < p_c \\ = 0 & \text{for } p > p_c \end{cases}$

  $\Rightarrow$ by continuity $\frac{1}{\xi(p_c)} = 0$.

- ξ is non-decreasing as it is a limit of non-decreasing functions

$\blacksquare$

<u>rmk</u> more precise estimates are known due to so-called Ornstein - Zernike theory : $\exists\, c = c(p) > 0$

s.t. $\mathbb{P}_p\left[0 \leftrightarrow ne_1\right] = c \cdot \frac{1}{n^{\frac{d-1}{2}}} e^{-\frac{1}{\xi(p)} n}\left(1+o(1)\right)$

as $n \to \infty$.