$\boxed{11}$ $p_c(\mathbb{Z}^2) = \frac{1}{2}$ $\quad$ (and $\theta(p_c) = 0$ on $\mathbb{Z}^2$)

- from now on, we restrict our attention to $\mathbb{Z}^2$.

thm (kesten 1980)
- $p_c(\mathbb{Z}^2) = \frac{1}{2}$
- $\mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \infty) = 0$.

thm (fitzner, hofstad 2017) for all $d \geq 11$, $\mathbb{P}_{p_c}[0 \leftrightarrow \infty] = 0$
& for $3 \leq d \leq 10$, the problem is open.

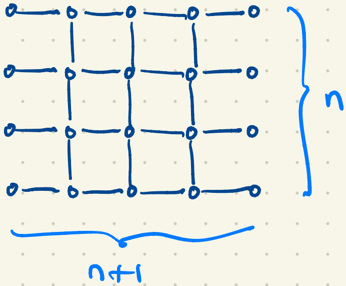
recall planar duality.

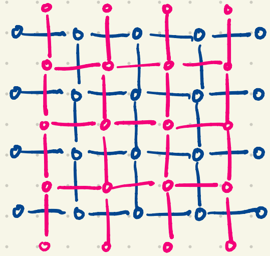
- $(\mathbb{Z}^2)^* :=$ the dual graph of $\mathbb{Z}^2$
  (vertices given by faces of $\mathbb{Z}^2$,
  $\{f, f'\}$ an edge of $(\mathbb{Z}^2)^*$ if they share an edge in $\mathbb{Z}^2$).

- $(\mathbb{Z}^2)^* \simeq \mathbb{Z}^2$ are isomorphic via shift by $(\frac{1}{2}, \frac{1}{2})$.
  ($\exists\, \phi : \mathbb{Z}^2 \to (\mathbb{Z}^2)^*$ bijection s.t. $\{v, v'\} \in E(\mathbb{Z}^2)$
  iff $\{\phi(v), \phi(v')\} \in E((\mathbb{Z}^2)^*)$).

- $\forall$ edges e in $\mathbb{Z}^2$, there corresponds exactly one dual edge e^* of $(\mathbb{Z}^2)^*$

- if $\omega \in \{0,1\}^E$, $\omega \sim \mathbb{P}_p$ then $\omega^* \in \{0,1\}^{E^*}$, $\omega^* \sim \mathbb{P}_{1-p}$ given by $\omega^*(e^*) = 1 - \omega(e)$.

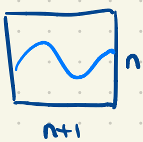
- let $G_n =$ 

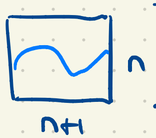
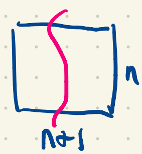
$G_n^* =$ 

then $G_n \cong G_n^*$ (G_n is self-dual)

lem $\exists$ a left-right crossing of G_n by ω

$\iff \nexists$ top-down crossing of G_n^* in ω^* ■

lem $\forall n \geq 1$, $\mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] = \frac{1}{2}$

proof by above $\mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] + \mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] = 1$

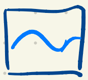
and by our work above, these two have same probability

■

proof of thm

$\boxed{p_c \leq \frac{1}{2}}$ assume $p > \frac{1}{2}$. then by sharpness, $\exists c > 0$

st. $\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial \Lambda_n] \leq e^{-cn}$.

however, $\frac{1}{2} = \mathbb{P}_{\frac{1}{2}}\Big[$  n, $n+1$ $\Big] \leq \sum_{\substack{x \text{ on} \\ \text{left side} \\ \text{of } G_n}} \mathbb{P}_{\frac{1}{2}}[x \leftrightarrow \partial \Lambda_n(x)]$

$= n e^{-cn}$ ⨳.

$\boxed{p_c \geq \frac{1}{2}}$ • assume $p_c < \frac{1}{2}$. then

$\mathbb{P}_{\frac{1}{2}}[\exists \infty \text{ cluster in } \omega] = 1$

but also $\mathbb{P}_{\frac{1}{2}}[\exists \infty \text{ cluster in } \omega^*] = 1$

$(\omega^* \sim \mathbb{P}_{\frac{1}{2}}$ too$)$

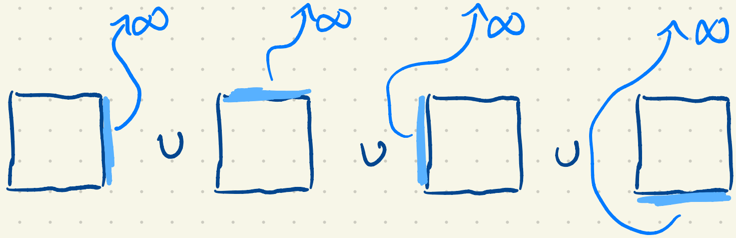
moreover, these ∞ clusters are unique.

• (zhang's argument) the above gives a contradiction.

fix $\varepsilon > 0$. pick $n \geq 1$ large enough st.

$\mathbb{P}_{\frac{1}{2}}[\Lambda_n \leftrightarrow \infty] \geq 1 - \varepsilon$

note:

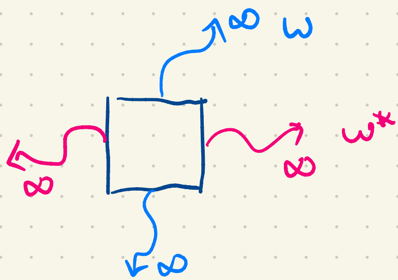
$\{\Lambda_n \leftrightarrow \infty\} =$ 

now by the square-root trick,

$$\mathbb{P}_{\frac{1}{2}}\left[\mathrm{top}(\Lambda_n) \overset{\mathbb{Z}^2 \setminus \Lambda_n}{\underset{\omega}{\longleftrightarrow}} \infty\right] \geq 1 - \varepsilon^{\frac{1}{4}}.$$

similarly for bottom (Λ_n), and similar for left (Λ_n) and right (Λ_n) using ω^* instead.

- by union bound,

$$\mathbb{P}_{\frac{1}{2}}\Big[$$  $$\Big] \geq 1 - 4\varepsilon^{1/4}$$

but on this event, either $\exists$ 2 ∞ primal clusters, or 2 ∞ dual clusters (joining top & bottom in primal prevents joining left & right in dual).

this contradicts uniqueness of ∞ clusters. $\#$

- hence $p_c = \frac{1}{2}$. moreover at p_c, we must have $\mathbb{P}_{\frac{1}{2}}[\infty \text{ cluster}] = 0$, otherwise just apply zhang's argument again. $\blacksquare$

rmk
- we only used $\mathbb{Z}^2 \simeq (\mathbb{Z}^2)^*$ and $\omega \sim \omega^*$, and FKG (sqrt trick), sharpness, and $\nexists\, \infty$ cluster; no independence used.

- on other 2D lattices G, one finds
$$p_c(G) + p_c(G^*) = 1$$

- on other lattices $\mathcal{L}$ in other models, one doesn't necessarily have self-dual point at $p = \frac{1}{2}$.

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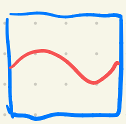
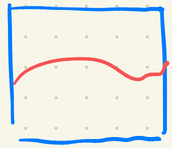
rmk in ex sheet 5, we show that $\Theta(p) > 0$

$\Rightarrow$

$\lim_{n \to \infty} \mathbb{P}_p\Big[$  $\Lambda_n \Big] = 1$

now since $\mathbb{P}_{\frac{1}{2}}\Big[$  $\Lambda_n \Big] \not\to 1$

one must have $p_c \geq \frac{1}{2}$; this is an alternate proof of part 2.