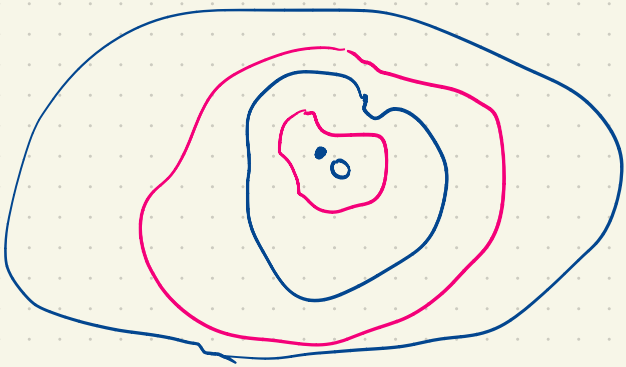
12. russo – seymour – welsh estimates

- what does the configuration look like at p_c?

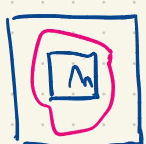
- we've shown $\theta(p_c) = 0$, so each point is surrounded by a dual circuit

  but also, $\nexists$ ∞ dual cluster, so each point (and each dual circuit) is surrounded by a primal circuit

  repeat this: one gets that $\exists$ ∞ many primal & dual circuits around each point.

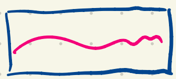


- goal of this chapter: $\exists$ circuits at every scale. more specifically:

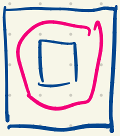
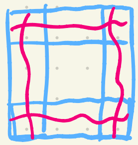
thm $\exists c > 0$ indep of n st. $\mathbb{P}_{\frac{1}{2}}$ [ Λ_{2n}] $\geq c > 0$ $\forall n \geq 1$.

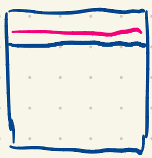
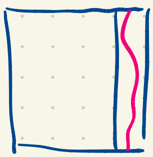
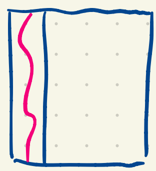
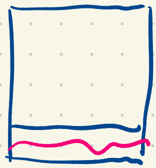
rmk the point of this thm (and the others in this chapter) is that c is independent of n.

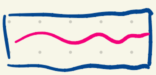
lem assume $\exists c > 0$ s.t. $\forall n \geq 1$,

$\mathbb{P}_{\frac{1}{2}}\Big[$  n $\Big] \geq c^{1/4}$. then **thm** holds.

4n

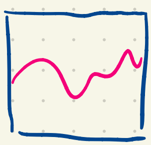
proof $\mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] \geq \mathbb{P}_{\frac{1}{2}}\Big[$  $\Big]$

$= \mathbb{P}_{\frac{1}{2}}\Big[$  $\cap$  $\cap$  $\cap$  $\Big]$

$\geq \mathbb{P}_{\frac{1}{2}}\Big[$  n $\Big]^4 \geq c$ $\blacksquare$

4n

FKG, rotation & translation invariance.

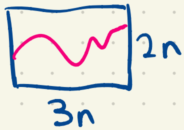
note we already have $\mathbb{P}_{\frac{1}{2}}\Big[$  n $\Big] = \frac{1}{2}$ $\forall n$.

n+1

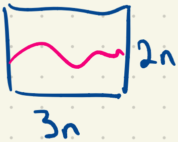
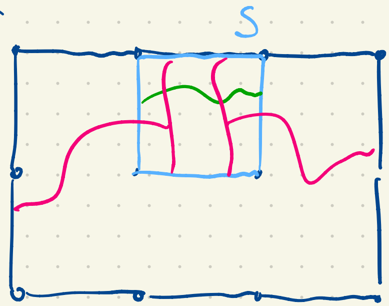
rmk we can use a square: replace $n+1$ by n, & get $\mathbb{P} \geq 1/2$.

rmk going from square to rectangle is the hard bit.

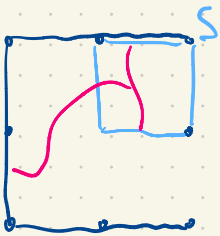
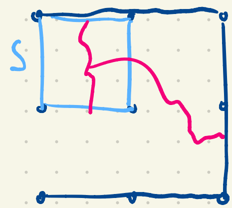
thm (russo 1978, seymour, welsh 1977)

$\exists\, c > 0$ st. $\forall n \geq 1$, $\mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] \geq c$.

proof **step 1**

$\mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] \geq \mathbb{P}_{\frac{1}{2}}\Big[$  $\Big]$

this event is B reflected

$= \mathbb{P}_{\frac{1}{2}}\Big[$  $\cap$  $\cap$  $\Big]$

this A

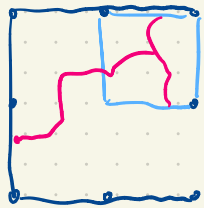
call this event B = { top-down crossing in S which is connected to LHS of big box $\boxed{3n}\,2n$ }

$$\underset{\text{FKG}}{\geq} \quad \mathbb{P}_{\frac{1}{2}}\big[B\big]^2\, \mathbb{P}_{\frac{1}{2}}\big[A\big] \geq \frac{1}{2}\, \mathbb{P}_{\frac{1}{2}}\big[B\big]^2 .$$

so suffices to prove $\exists\, c' > 0$ st. $\forall n \geq 1$,

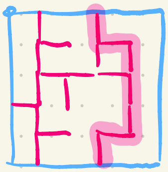
$$\mathbb{P}_{\frac{1}{2}}\big[B\big] \geq c' .$$

step 2

$B =$ 

- let Γ be the right-most vertical crossing of $\square =: S$ (Γ is random).

eg



$\sim$ this is Γ

S

- let γ be a fixed path from top to bottom of S.

consider the event $\{\Gamma = \gamma\}$. we can write this as

$$\{\Gamma = \gamma\} = \{\text{edges on } \gamma \text{ are open}\}$$
$$\cap \{\text{no crossing to the right of } \gamma \text{ in } S\}$$

in particular, $\{\Gamma = \gamma\}$ is only dependent on edges on γ & to the right of γ inside S.

- so, $$\mathbb{P}_{\frac{1}{2}}[B] = \mathbb{P}_{\frac{1}{2}}\Big[\bigcup_{\gamma \subset S} B \cap \{\Gamma = \gamma\}\Big]$$

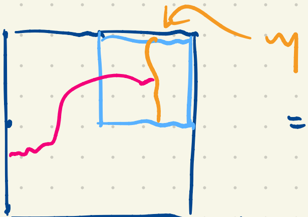
disjoint events

$$= \sum_{\gamma \subset S} \mathbb{P}_{\frac{1}{2}}\big[B \cap \{\Gamma = \gamma\}\big]$$

$$= \sum_{\gamma \subset S} \mathbb{P}_{\frac{1}{2}}\left[\Gamma \text{ exists and } = \gamma\right] \cdot \qquad \boxed{*}$$

$$\cdot\, \mathbb{P}_{\frac{1}{2}}\left[B \;\middle|\; \Gamma \text{ exists and } = \gamma\right]$$

$$\mathbb{P}_{\frac{1}{2}}\left[B \;\middle|\; \Gamma \text{ exists and } = \gamma\right]$$

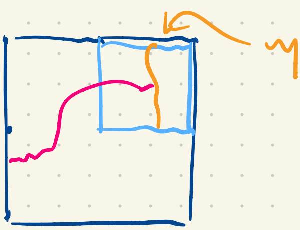
$\geq \mathbb{P}_{\frac{1}{2}}\Big[$ 

$= \{\gamma \leftrightarrow \text{LHS of large box}\}$

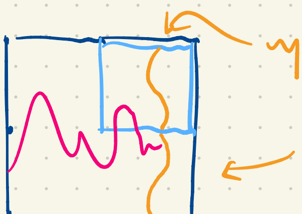
$\Big| \; \Gamma \text{ exists and } = \gamma \Big]$

but now these are independent, so

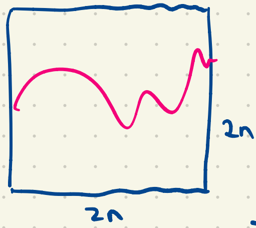
since $\{\Gamma = \gamma\}$ depends on γ & edges right of γ, & $\{\gamma \leftrightarrow \text{LHS}\}$ depends on edges left of γ

$= \mathbb{P}_{\frac{1}{2}}\Big[$ 

$\Big]$

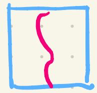
$\geq \frac{1}{2}\mathbb{P}_{\frac{1}{2}}\Big[$ 

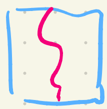
$= \{\text{LHS} \leftrightarrow \gamma \text{ or reflection of } \gamma\}$ $\Big]$

$$\geq \frac{1}{2}\,\mathbb{P}_{\frac{1}{2}}\Bigg[$$


$$\Bigg] \geq \frac{1}{4}.$$

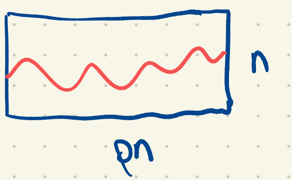
- plugging back into $\boxed{*}$, we have

$$\mathbb{P}_{\frac{1}{2}}[B] \geq \frac{1}{4} \sum_{\gamma \subset S} \mathbb{P}_{\frac{1}{2}}\Big[$$

$$\cap \{\Gamma = \gamma\}\Big]$$

$$= \frac{1}{4}\,\mathbb{P}_{\frac{1}{2}}\Big[$$

$$\Big] \geq \frac{1}{8} \qquad \blacksquare$$

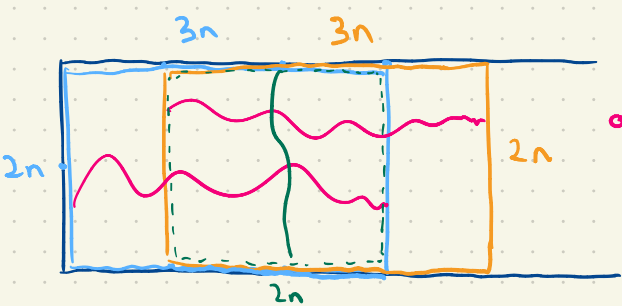
corollary (box-crossing)

let $\rho > 0$. $\exists\, c = c(\rho) > 0$ st. $\forall n \geq 1$,

$$\mathbb{P}_{\frac{1}{2}}\Bigg[$$


$$\Bigg] \geq c.$$

proof



and repeat $\longrightarrow$

& use FKG $\blacksquare$

note this proves lemma & thm from start of section.

cor (one arm exponent)

$\exists c>0$ st. $\forall k \geq 1$, $\frac{1}{2k} \leq \theta_k(\frac{1}{2}) \leq \frac{1}{k^c}$

where $\theta_k(p) = \mathbb{P}_p[0 \leftrightarrow \partial \Lambda_k]$.

proof: exercise sheet 5.

conjecture $\exists C>0$ st. $\theta_k(p) \sim C k^{-5/48}$ as $k \to \infty$.

rmk this is proved for site percolation on the triangular lattice (smirnov, werner 2001).