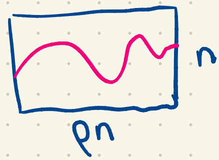
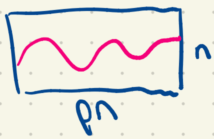
13 conformal invariance

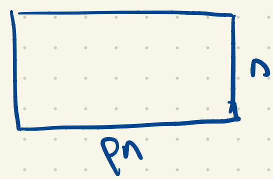
13.1 the result

- in the previous chapter, we saw that $\exists\, c > 0$ s.t.

$$1-c \geq \mathbb{P}_{\frac{1}{2}}\Bigg[$$  $$\Bigg] \geq c \qquad \forall n \geq 1$$

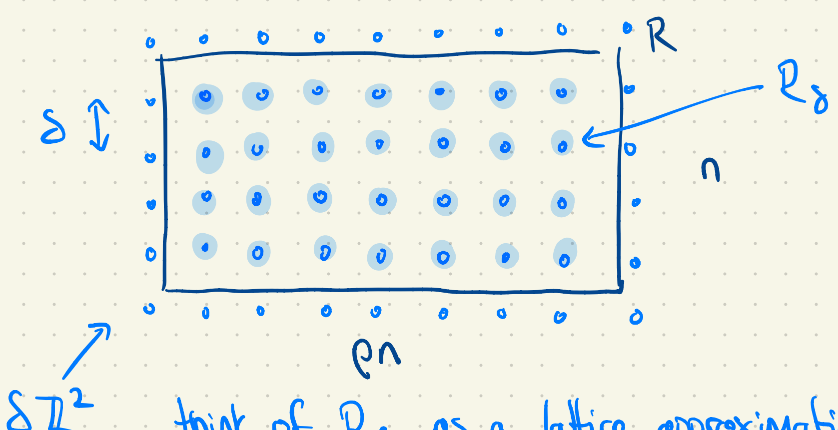
this is a sign of a stronger result: that

(*) $\mathbb{P}_{\frac{1}{2}}\Bigg[$  $\Bigg]$ has a limit $\in (0,1)$ as $n \to \infty$.

- we can rephrase this. let $R =$ 

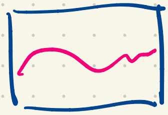
be the continuous rectangle in $\mathbb{R}^2$.

let $\delta > 0$. let $R_\delta = \delta\mathbb{Z}^2 \cap R$

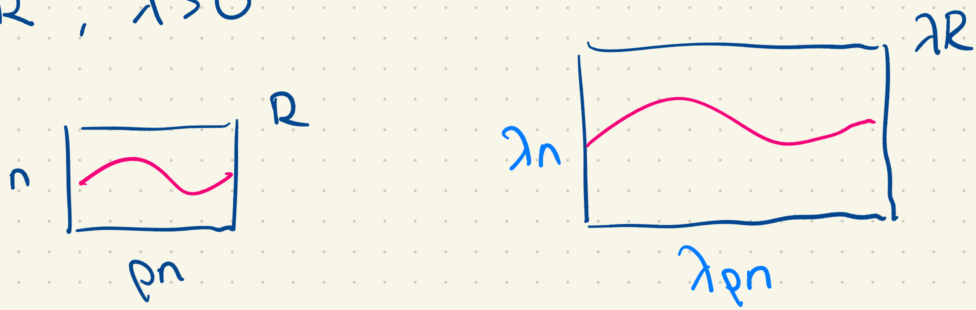


think of R_δ as a lattice approximation to R

- instead of making the rectangle larger, we can make δ smaller. so $(*)$ can be rephrased as

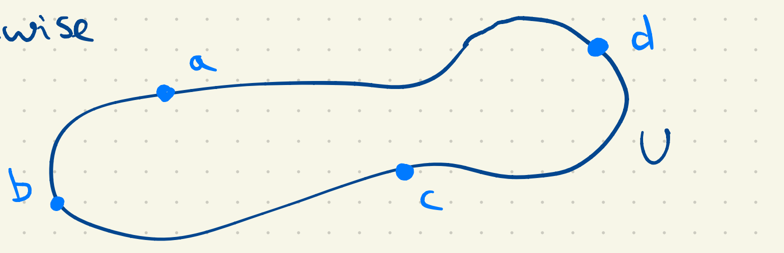
$\lim_{\delta \to 0} \mathbb{P}_{\frac{1}{2}}\Big[$  $R_\delta \Big]$ exists.

- moreover, this will be independent of the scale of R, that is, one gets the same limit for λR, $\lambda > 0$



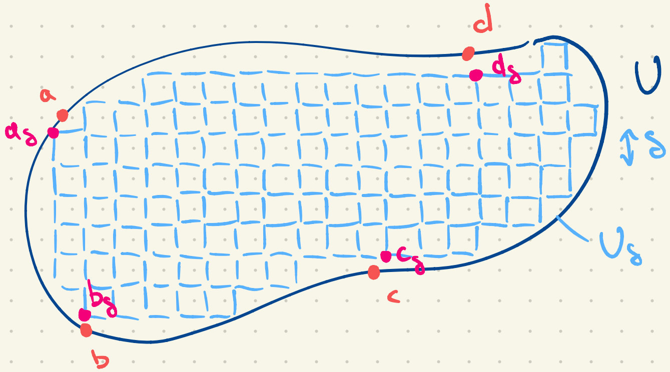
we call this scale invariance. in fact, something stronger is true — conformal invariance.

def a topological rectangle is a quintuple (U, a, b, c, d) where U is a bounded Jordan domain (interior of a continuous, simple curve), and $a, b, c, d \in \partial U$ are four distinct points on U's boundary, counterclockwise

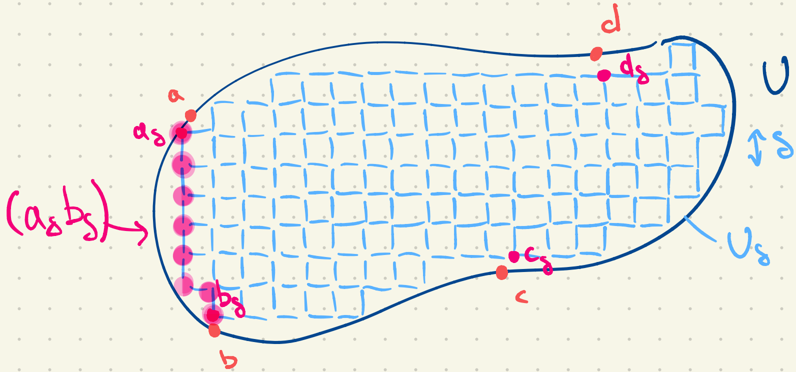


we write (a,b) for the counterclockwise arc from a to b in ∂U.

- for a topological rectangle (U, a, b, c, d), $\delta > 0$, write $(U_\delta, a_\delta, b_\delta, c_\delta, d_\delta)$ for the discrete approximation:
  - $U_\delta := \delta\mathbb{Z}^2 \cap U$
  - $a_\delta :=$ closest vertex of U_δ to a, $b_\delta, c_\delta, d_\delta$ similar.



- similar to the continuous case, write $(a_\delta b_\delta)$ for the set of vertices in ∂U_δ on the counterclockwise arc from a_δ to b_δ.



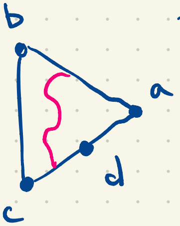
prediction from physics: let (U, a, b, c, d), (U', a', b', c', d') be "conformally equivalent" (ie. $\phi: U \to U'$ s.t. $\phi(a) = a'$, etc). then for small δ,

ϕ conformal

$$\lim_{\delta \to 0} \mathbb{P}_{\frac{1}{2}}\left[(a_\delta b_\delta) \overset{U_\delta}{\longleftrightarrow} (c_\delta d_\delta)\right] = \lim_{\delta \to 0} \mathbb{P}_{\frac{1}{2}}\left[(a'_\delta b'_\delta) \overset{U'_\delta}{\longleftrightarrow} (c'_\delta d'_\delta)\right]$$

this is conformal invariance (we haven't defined yet what conformal maps are — this is on the next page).

- in 1992, langlands, pouliot and saint-aubin gave numerical evidence for this.
- later in 1992, cardy predicted an explicit formula, and carleson noticed it has a simple form when $U = T$, T an equilateral triangle.

$\mathbb{P}\Bigg[$  $\Bigg] = x := \dfrac{|d-c|}{|a-c|}$

- in 2001 smirnov proved cardy's formula, but only for site percolation on the triangular lattice. this was a huge result, and was a big part of why smirnov got a fields medal in 2010.

detour into complex analysis

def
- let $U \subseteq \mathbb{C} \cong \mathbb{R}^2$ be a **domain** (connected open set).
- a function
$$\phi : U \to \mathbb{C}$$
is **holomorphic at $z \in U$** if $\lim_{\substack{\varepsilon \in \mathbb{C} \\ |\varepsilon| \to 0}} \frac{\phi(z+\varepsilon) - \phi(z)}{\varepsilon}$ exists.
- we denote this limit by **$\phi'(z)$**.
- ϕ is **holomorphic on U** if it is holomorphic at all $z \in U$

thm (morera)

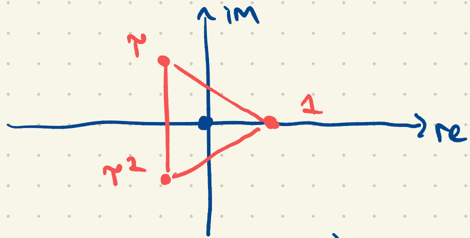
let $\phi : U \to \mathbb{C}$ continuous. ϕ is holomorphic on U iff all closed contour integrals of ϕ vanish, that is:
$$\oint_\Gamma \phi(z)\, dz = 0$$
for every piecewise C^1 closed curve Γ in U, which is contractable in U.

def let $U, U' \subseteq \mathbb{C}$ be two domains. a bijection $\phi : U \to U'$ is called a conformal equivalence

if
- ϕ is holomorphic on U
- ϕ^{-1} is holomorphic on U'.

(in fact ϕ^{-1} holo on U' follows from ϕ bijective and holomorphic)

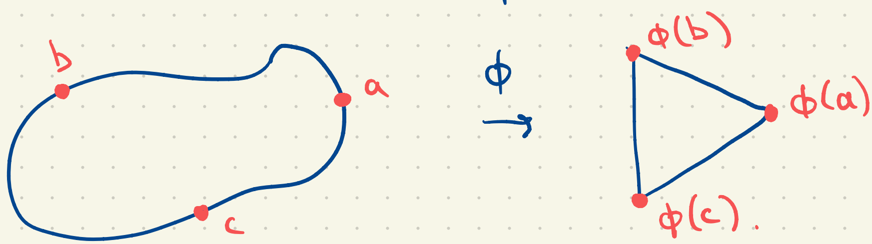
def let T denote the domain given by the (open) triangle in $\mathbb{C}$ with vertices 1, $\tau = e^{2\pi i/3}$, τ^2



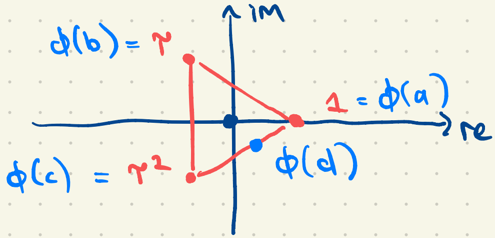
thm (riemann mapping theorem)
let $U \subseteq \mathbb{C}$ be a simply connected domain.
$\exists$ conformal equivalence $\phi : U \to T$. if $z \in U$ is fixed, then ϕ is unique if $\phi(z) = 0$, $\phi'(z) \in \mathbb{R}_{>0}$

thm (carathéodory) let U be a jordan domain. then $\phi : U \to T$ above extends to a homeomorphism $\phi : \overline{U} \to \overline{T}$. ← (closure)

thm (riemann part 2): for U jordan, if $a, b, c \in \partial U$ ordered anticlockwise, then $\exists$ a unique $\phi : \overline{U} \to \overline{T}$ conformal s.t. $\phi(a) = 1$, $\phi(b) = \tau$, $\phi(c) = \tau^2$.



rmk if (U,a,b,c,d) is a topological rectangle, and $\phi: U \to T$ conformal such that $\phi(a)=1$, $\phi(b)=\tau$, $\phi(c)=\tau^2$, then $\phi(d)$ is some point on the line from τ^2 to 1



def let $x := \dfrac{|\phi(d)-\phi(c)|}{|\phi(a)-\phi(c)|} = \dfrac{|\phi(d)-\tau^2|}{|1-\tau^2|}$

thm (smirnov 2001) using site percolation on the triangular lattice in our definitions above, $\forall$ topological rectangles (U,a,b,c,d)

$$\lim_{\delta\to 0} \mathbb{P}_{\frac{1}{2}}\left[(a_\delta b_\delta) \overset{U_\delta}{\longleftrightarrow} (c_\delta d_\delta)\right] = x.$$

rmk we'll study a more recent proof by Kristoforov.

rmk how to prove this? we will define a discrete function

$$F_\delta : U_\delta \to T,$$

which is :

- "discrete holomorphic"
- which tends $F_\delta \to \phi$ as $\delta \to 0$
- and which contains the information

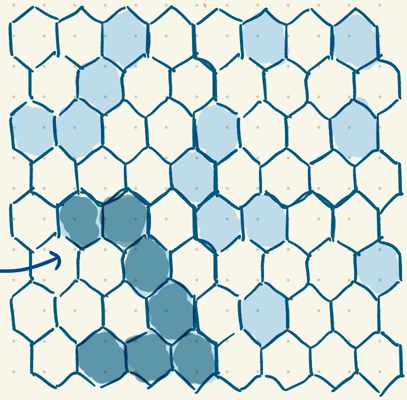
$$\mathbb{P}_{\frac{1}{2}}[(a_\delta b_\delta) \leftrightarrow (c_\delta d_\delta)]$$

13.2 site percolation on triangular lattice & loop representation

- let $\mathbb{H}$ be the hexagonal lattice, the dual of the triangular lattice.
- we consider a percolation process $\mathbb{P}_p$ on faces of $\mathbb{H}$ denoted $F(\mathbb{H})$ (equivalent to vertices of triangular lattice)

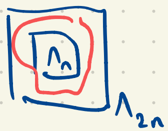
ie: $\Omega = \{0,1\}^{F(\mathbb{H})}$, $\mathcal{F} = \sigma(\text{cylinder sets})$, etc.
we define a cluster of $\omega \in \Omega$ to be a maximal connected component of open faces.

a cluster



thm RSW estimates hold for $\mathbb{P}_{\frac{1}{2}}$:

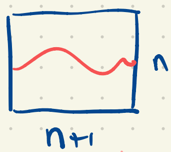
$\exists c > 0$ st. $\forall n \geq 1$,

$\mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] \geq c$

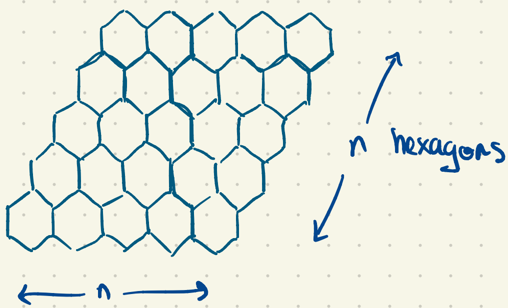
and consequently, $\exists c' > 0$ st. $\forall k \geq 1$:

$$\frac{1}{4k} \leq \mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial \Lambda_k \right] \leq \frac{1}{k^c}$$

proof the crucial ingredient for bond percolation on $\mathbb{Z}^2$ was:

$\mathbb{P}^{\text{bond } \mathbb{Z}^2}_{\frac{1}{2}}\Big[$  $\Big] = \frac{1}{2}$

on $\mathbb{H}$, let Λ_n be



in place of the dual ω^*, we use the closed faces $1-\omega$:

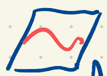
we have :

$\exists$ top-down crossing of Λ_n in ω

$\Leftrightarrow$

$\nexists$ left-right crossing of Λ_n in $1-\omega$, and by symmetry, at $p = \frac{1}{2}$, these have the same probability.

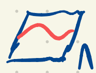
so

$$\mathbb{P}_{\frac{1}{2}}\Big[$$  $$_{\Lambda_n}\Big] = \frac{1}{2}.$$

further, FKG holds too.

the rest of the proof follows the same as in bond percolation on $\mathbb{Z}^2$.

■

thm one can check that monotonicity, ergodicity, FKG, sharpness and $\exists \leq 1$ ∞ cluster all hold for $\mathbb{P}_p$. along with $\mathbb{P}_{\frac{1}{2}}\Big[$  $_{\Lambda_n}\Big] = \frac{1}{2}$, we can rerun the proof that $p_c = \frac{1}{2}$ and $\theta(p_c) = 0$.

---

rmk $\mathbb{P}_{\frac{1}{2}}$ on a finite subgraph G of $\mathbb{H}$ is the uniform measure on $\Omega_G = \{0,1\}^{F(G)}$. indeed,

$$\mathbb{P}_{\frac{1}{2}}[\omega] = \frac{1}{2}^{o(\omega)} \cdot \frac{1}{2}^{c(\omega)} = \frac{1}{2}^{|F(G)|}$$

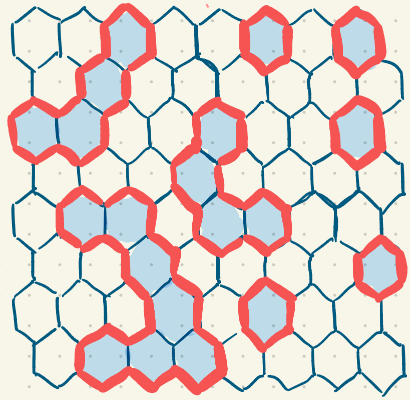
where $F(G)$ is the set of faces of G.

loop representation

def G is a $\mathbb{H}$-domain if it is a subgraph of $\mathbb{H}$ obtained by gluing faces of $\mathbb{H}$ together.

- consider our percolation process on a $\mathbb{H}$-domain G. a percolation configuration (a subset of the faces of G) gives

a loop configuration on $\mathcal{G}$:
(a subgraph of $\mathcal{G}$ such that every vertex has even degree)



- let $\Omega_{\mathcal{G}}^{loop} = \{$loop configs on $\mathcal{G}\}$
- we formalise the above:

$$\alpha : \Omega_{\mathcal{G}} = \{0,1\}^{F(\mathcal{G})} \longrightarrow \Omega_{\mathcal{G}}^{loop}$$

as

$e \in \alpha(\omega)$ iff $\begin{cases} \omega \text{ differs on faces either side of } e & e \text{ in bulk of } \mathcal{G} \\ \omega = 1 \text{ on face next to } e & e \text{ on boundary} \end{cases}$

- this map has an inverse:
given $\eta \in \Omega_{\mathcal{G}}^{loop}$, fix the "outside face" of $\mathcal{G}$ to be closed. then define $\omega \in \Omega_{\mathcal{G}}$ such that ω changes between open & closed across an edge e iff $e \in \eta$.

- we now have a measure $\mathbb{P}_{\frac{1}{2}}$ on $\Omega_{\mathcal{G}}^{loop}$, the uniform measure:

$$\mathbb{P}_{\frac{1}{2}}[\eta] = \frac{1}{2^{F(\mathcal{G})}} \quad \forall\ \eta \in \Omega_{\mathcal{G}}^{loop}.$$

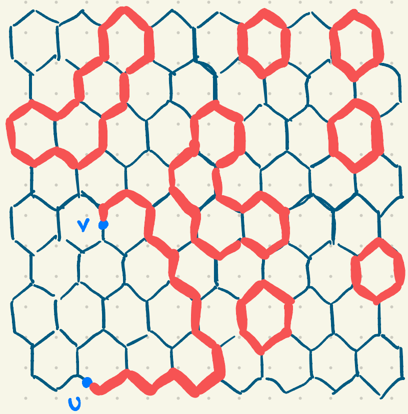
<u>rmk</u> recall our goal is to define a "discrete holomorphic" function $F_\delta : U_\delta \to T$ st. $F_\delta \to \phi$ as $\delta \to 0$, and F contains the information
$\mathbb{P}_{\frac{1}{2}}[(a_\delta b_\delta) \leftrightarrow (c_\delta d_\delta)]$.

- our function F_δ will actually not be defined on the vertices of U_δ, but on the midpoints of its edges.

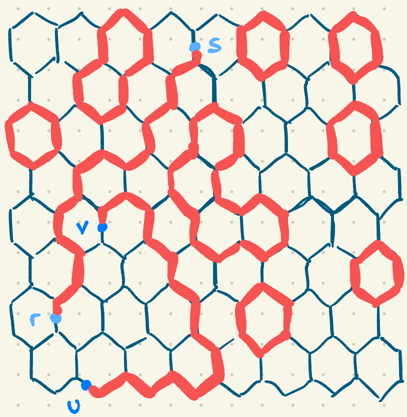
def let $E_{mid} = E_{mid}(U_\delta)$ be the midpoints of the edges of U_δ, where U_δ is a $\delta\mathbb{H}$-domain approximating $U \subseteq \mathbb{C}$.

- we also need a slight modification of Ω_G^{loop}.

def • let $u, v \in E_{mid}(G)$. let $\Omega_{G,uv}^{loop}$ be the set of configurations of loops <u>and</u> a self-avoiding path $u \leftrightarrow v$ (loops & the path all disjoint)



- similarly for mid-edges $u, v, r, s \in E_{mid}(G)$ let $\Omega_{G,uv,rs}^{loop}$ be the same with a self avoiding path $r \leftrightarrow s$ too.



- <u>rmk</u> : $\Omega_{G,uv,rs}^{loop} = \Omega_{G,uv}^{loop}$ when $r = s$

- <u>rmk</u> : $\Omega_{G,uv,rs}^{loop} = \emptyset$ when $u = r$.

- we can now define our F_δ.

13.3 the function F_δ (the fermionic observable)

<u>def</u> let G be a $\mathbb{H}$-domain, let $a, b, c \in E_{mid}(G)$ be midedges on the boundary of G. recall $\tau = e^{2\pi i/3}$.

- let
$$F_a : E_{mid}(G) \to \mathbb{R}$$
$$F_a(z) = \frac{|\Omega^{loop}_{G, az, bc}|}{|\Omega^{loop}_G|} = \frac{1}{2|F(G)|} |\Omega^{loop}_{G, az, bc}|$$

& similar for F_b, F_c.

- let
$$F : E_{mid}(G) \to \mathbb{C}$$
as $F(z) = F_a(z) + \tau F_b(z) + \tau^2 F_c(z)$.

<u>rmk</u> this F is known as a <u>fermionic observable</u>

<u>lem</u> $F(a) = 1$, $F(b) = \tau$, $F(c) = \tau^2$, and F maps $E_{mid}(G)$ to T.

<u>proof</u> firstly we show $F(a) = 1$ (& similarly $F(b) = \tau$, $F(c) = \tau^2$). indeed,

- $$F(a) = \frac{1}{2|F(G)|}\left[|\Omega^{loop}_{aa, bc}| + |\Omega^{loop}_{ab, ac}| + |\Omega^{loop}_{ac, ab}| \right]$$
$$= \frac{1}{2|F(G)|}\left[|\Omega^{loop}_{bc}| + 0 + 0 \right].$$

- for η_1, η_2 some subsets of half-edges of G, let $\eta_1 \Delta \eta_2$ be the symmetric difference of η_1, η_2;

$$\eta_1 \Delta \eta_2 := (\eta_1 \setminus \eta_2) \cup (\eta_2 \setminus \eta_1).$$

- we want $|\Omega^{loop}_{bc}| = |\Omega^{loop}| = 2^{|F(G)|}$.
  fix some $\eta_0 \in \Omega^{loop}_{bc}$. define a map

$$\pi : \Omega^{loop}_{bc} \to \Omega^{loop}$$
as $$\pi(\eta) = \eta \Delta \eta_0$$

π is a bijection, so claim holds. similar for $F(b), F(c)$.

---

secondly, we show $F_a(z) + F_b(z) + F_c(z) = 1$, which gives the result. well,

$$LHS = \frac{1}{|\Omega^{loop}|} \Big| \underbrace{\Omega^{loop}_{az,bc} \cup \Omega^{loop}_{bz,ac} \cup \Omega^{loop}_{cz,ab}}_{\text{call this } \Omega^*} \Big|$$

let $\eta_0 \in \Omega^*$ fixed. let $\pi : \Omega^* \to \Omega^{loop}$ as

$$\pi(\eta) = \eta \Delta \eta_0 .$$ π is a bijection. ∎

3.4 the proof

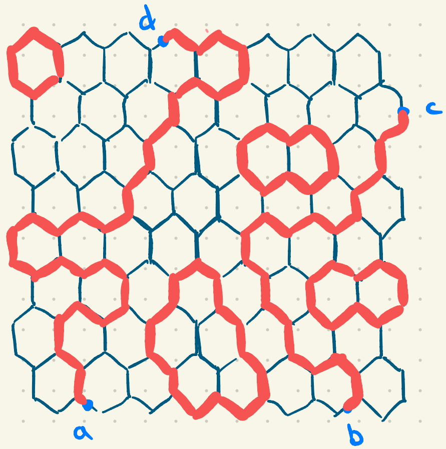
- we have to prove 3 properties of F to complete the proof:

  1. F contains the information $\mathbb{P}_{\frac{1}{2}}[(ab) \leftrightarrow (cd)]$
  2. F is "discrete holomorphic"
  3. on $U_\delta \to U \subset \mathbb{C}$, $F_\delta \to \phi$, where $\phi : U \to T$ conformal.

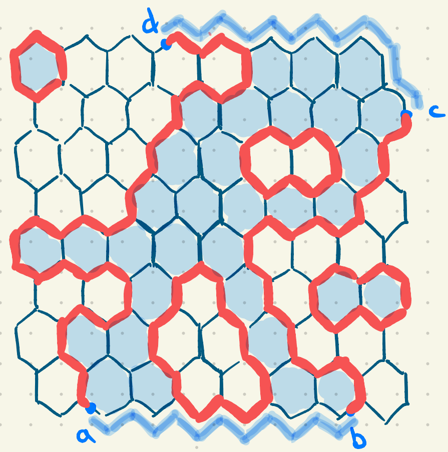
let's do 1.

- let G be a $\mathbb{H}$-domain. let $a, b, c, d \in E_{mid}(G)$ be boundary mid-edges, ordered counterclockwise.

  then $F_a(d) = \mathbb{P}_{\frac{1}{2}}\left[(ab) \leftrightarrow (cd)\right]$



$\to$



- earlier, we defined a map $\Omega_G^{loop} \to \Omega_G$ where we set the "external face" to be closed, & then ω changes open $\leftrightarrow$ closed across $e \in E(G)$ iff $e \in \eta$.
- define the same map, except the "external face" is closed on the arcs (bc), (da), and open on (ab), (cd).

- this map bijects $\Omega^{\text{loop}}_{ad,bc} \leftrightarrow \{(ab) \leftrightarrow (cd)\}$;

  now $F_a(d) = \dfrac{1}{|\Omega^{\text{loop}}_G|} |\Omega^{\text{loop}}_{G,ad,bc}| = \dfrac{1}{|\Omega_G|} |\{(ab) \leftrightarrow (cd)\}| = \text{RHS},$

- similarly, prove $F_c(d) = \mathbb{P}_{\frac{1}{2}}\left[(ad) \leftrightarrow (bc)\right]$
  $= 1 - \mathbb{P}_{\frac{1}{2}}\left[(ab) \leftrightarrow (cd)\right]$

  and $F_b(d) = 0$.

to finish,

$$F(d) = \mathbb{P}_{\frac{1}{2}}\left[(ab) \leftrightarrow (cd)\right] + \tau^2 \left(1 - \mathbb{P}_{\frac{1}{2}}\left[(ab) \leftrightarrow (cd)\right]\right)$$

$\boxed{**}$
$$= \tau^2 + (1 - \tau^2)\, \mathbb{P}_{\frac{1}{2}}\left[(ab) \leftrightarrow (cd)\right] \qquad \blacksquare$$

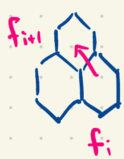
so $\mathbb{P}_{\frac{1}{2}}\left[(ab) \leftrightarrow (cd)\right]$

---

now ②: F is discrete holomorphic.

- we'll prove an analogy of the condition given by Morera's thm.

def
- γ is a loop on $F(G)$ if γ is a sequence $e_1, \ldots, e_n$, $e_i \in E(G^*)$, $e_i = \{f_i, f_{i+1}\}$, $f_{n+1} = f_1$ and $f_1, \ldots, f_n$ all distinct.
- for $e_i = \{f_i, f_{i+1}\}$ in a loop, we interpret $f_{i+1} - f_i$ as an element of $\mathbb{C}$ given by the vector $f_i \to f_{i+1}$.



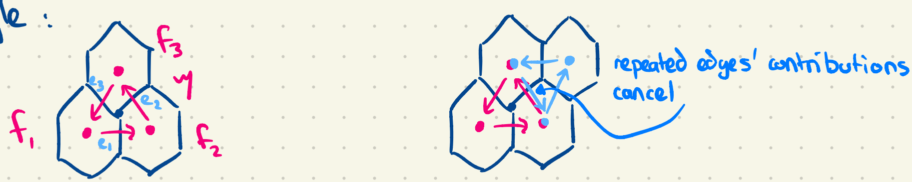
$f_{i+1} - f_i = \tau \in \mathbb{C}$

lem for all loops $\gamma = \{e_1, \ldots, e_n\}$ in $F(G)$,

$$\oint_\gamma F(z)\,dz := \sum_{i=1}^{n} F(e_i^{mid}) \cdot (f_{i+1} - f_i) = 0.$$

where e_i^{mid} is e_i's midpoint.

proof • it suffices to prove lem for $\gamma = \{e_1, e_2, e_3\}$ a triangle:



indeed, one can form any loop as the boundary of a union of such triangles; then sum the result over all those triangles.

• we show that

$$F(e_1) + \tau F(e_2) + \tau^2 F(e_3) = 0$$

$$\begin{aligned} LHS = \frac{1}{2^{F(G)}} \Big[& |\Omega^{loop}_{ae_1, bc}| + \tau |\Omega^{loop}_{be_1, ac}| + \tau^2 |\Omega^{loop}_{ce_1, ab}| \\ & + \tau \left(|\Omega^{loop}_{ae_2, bc}| + \tau |\Omega^{loop}_{be_2, ac}| + \tau^2 |\Omega^{loop}_{ce_2, ab}| \right) \\ & \tau^2 \left(|\Omega^{loop}_{ae_3, bc}| + \tau |\Omega^{loop}_{be_3, ac}| + \tau^2 |\Omega^{loop}_{ce_3, ab}| \right) \Big] \end{aligned}$$

consider $\bigcup_{i=1}^{3} \left(\Omega^{loop}_{ae_i, bc} \cup \Omega^{loop}_{be_i, ac} \cup \Omega^{loop}_{ce_i, ab} \right)$

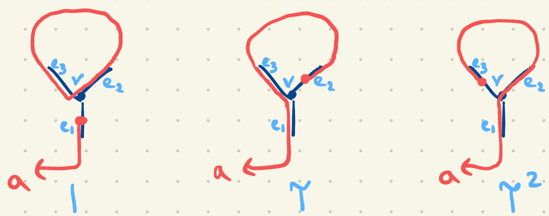
each element of this set gets a coefficient 1, τ, or τ^2. if the number of each is the same, then as $1 + \tau + \tau^2 = 0$, the result follows.

- split this set into triples which coincide outside of e_1, e_2, e_3 :

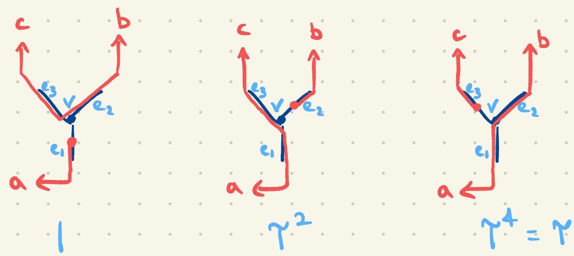
boundary to boundary

triple type 1 :
- the self-avoiding path $\partial \leftrightarrow \partial$ <u>does not</u> pass thru v
- ≤ 3 half-edges of e_1, e_2, e_3 are in the configuration

triple type 2 :
- the self-avoiding path $\partial \leftrightarrow \partial$ <u>does not</u> pass thru v
- > 3 half-edges of e_1, e_2, e_3 are in the configuration



triple type 3 :
- the self-avoiding path $\partial \leftrightarrow \partial$ <u>does</u> pass thru v



result follows from $1 + \tau + \tau^2 = 0$ and summing over all triples. ■

lastly, part 3 : $F_\delta \to \phi$.

method:

1. we can extend F_δ in some simple way to a function on $\overline{U}$ (not just U_δ).

2. the RSW estimates imply that F_δ is Hölder : $\exists\, \alpha \in (0,1]$, $C>0$ st. uniformly in δ, $\forall\, z, z' \in \overline{U}$,

$$|F_\delta(z) - F_\delta(z')| \leq C|z-z|^\alpha$$

3. this, along with F_δ uniformly bounded, gives $\{F_\delta\}_{\delta>0}$ precompact:

for all sequences $\delta_k \to 0$, $\exists$ subsequence δ_{k_l} st.

$$F_{\delta_{k_l}} \to F$$

uniformly on $\overline{U}$

4. we now have many limits (one for each sequence $\delta_k \to 0$); it remains to show they are the same and $= \phi$:
   - F_δ discrete holo $\Rightarrow$ F holomorphic
   - $F(a)=1$, etc and $F((a,b)) = (1,\tau)$
     $\Rightarrow$ F homeomorphism on $\partial U \to \partial T$

$\Rightarrow$ by riemann mapping thm, $F = \phi$.

1

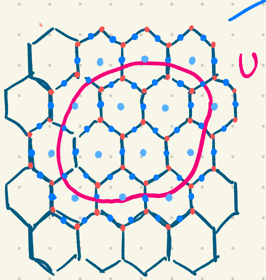
- let (U, a, b, c, d) be a topological rectangle, and let $(U_\delta, a_\delta, b_\delta, c_\delta, d_\delta)$ be an approximating sequence of $\mathbb{H}$-domains. F_δ is defined on $E_{mid}(U_\delta)$ as F above, and maps

$$F_\delta : E_{mid}(U_\delta) \to T$$

with $F_\delta(a_\delta) = 1$, $F_\delta(b_\delta) = \tau$, $F_\delta(c_\delta) = \tau^2$.

- we extend F_δ to a piecewise-linear function on $\overline{U}$:
first for all mid-edges or mid-faces of $\delta\mathbb{H}$ intesecting $\overline{U}$, or vertices of these hexagons,
let F_δ be equal to F_δ at the closest element of $E_{mid}(U_\delta)$;
if there's more than one closest, pick one of them arbitrarily.

this defines F_δ on the vertices of a triangulation covering $\overline{U}$. extend to the interior of each triangle linearly. lastly, restrict to $\overline{U}$.



all vertices
+
all mid-edges
+
all mid-faces
of hexagons intersecting U.

in particular, $F_\delta(\partial U) \subset \partial T$.

$\boxed{3}$ thm from analysis : arzelà-ascoli :

if $F_\delta : \bar{U} \to \mathbb{C}$ is uniformly bounded and uniformly equicontinuous, then for all subsequences F_{δ_k} $\exists$ a further subsequence $F_{\delta_{k_\ell}}$ which converges uniformly.

- F_δ uniformly bounded if $\exists M > 0$ st. $\forall x \in \bar{U}$, $\forall \delta > 0$, $|F_\delta(x)| < M$.
- F_δ uniformly equicontinuous if $\forall \varepsilon > 0$, $\exists \lambda > 0$ st.

$$|x-y| < \lambda \implies |F_\delta(x) - F_\delta(y)| < \varepsilon$$

for all $x, y \in \bar{U}$, $\forall \delta > 0$.

def F_δ is Hölder continuous if $\exists M > 0$, $\alpha > 0$ st.

$$|F_\delta(x) - F_\delta(y)| < M \cdot |x-y|^\alpha$$

lem if F_δ is Hölder continuous for $\alpha \leq 1$, then F_δ is uniformly equicontinuous.

$\boxed{2.}$

lem the RSW theorem for site percolation on the triangular lattice $\Rightarrow$ F_δ is Hölder with $\alpha \leq 1$.

done on ex sheet 6 (special case)

- by arzelà ascoli, for all subsequences F_{δ_k}, $\exists$ a further subsequence $F_{\delta_{k_\ell}}$ uniformly convergent on $\bar{U}$

$$F_{\delta_{n_k}} \to F$$

- we see that $F(a) = 1$, $F(b) = \tau$, $F(c) = \tau^2$

and by $\boxed{**}$, F maps ∂U to ∂T. F is also continuous, so it is surjective.

$\boxed{4.}$

- let γ be any rectangular contour in U, and let γ_{δ_n} be the loop on faces of $\delta_n \mathbb{H}$ of maximal area lying inside γ. then

F cts and bounded

$$\oint_\gamma F(z)\,dz = \lim_{n\to\infty} \oint_{\gamma_{\delta_n}} F(z)\,dz$$

uniform convergence

$$= \lim_{n\to\infty} \lim_{l\to\infty} \oint_{\gamma_{\delta_n}} F_{\delta_{k_l}}(z)\,dz = 0$$

as $F_{\delta_{k_l}}$ is discrete holo.

hence F is holomorphic on U.

- the argument principle now gives that $F = \phi$. indeed, it suffices to show F injective; we know F is surjective, & a bijective holomorphic map has holomophic inverse. F is then already extended to ∂U, & agrees with ϕ on 3 points on ∂U, so $F = \phi$ by riemann part 2.

  assume $F: U \to T$ not injective; $\exists\, u, v \in U$ s.t. $F(u) = F(v) = w \in T$. define $G(z) = F(z) - w$. then G is a holomorphic, surjective map $\bar{U} \to T - w$ with at least two zeros, u, v, in U.

however, we showed above that F maps (a,b) to $(1,\gamma)$, etc. this means F has winding number 1 along ∂U. more precisely, for γ_k a sequence of simple curves approaching ∂U,

$$\lim_{k\to\infty} \oint_{\gamma_k} \frac{F'(z)}{F(z)}\,dz = 1.$$

the same holds for G. however, the argument principle says that

$$\oint_{\gamma_k} \frac{G'(z)}{G(z)}\,dz = \#\text{zeros of } G \text{ in } \gamma_k - \#\text{poles of } G \text{ in } \gamma_k$$

(with multiplicities)

γ_k must eventually contain u and v, and G is bounded, so the limit above must be ≥ 2. contradiction, so F must be injective.

- every subsequence of F_δ has a subsequence converging uniformly to ϕ on $\overline{U}$; so $F_\delta \to \phi$ uniformly.

$\blacksquare$