14. beyond bernoulli percolation

- there are many generalisations of and similar models to bernoulli percolation.

- we'll introduce one widely studied example: FK percolation (aka the random cluster model).

- this is a natural generalisation, but is also important because it is closely linked to the (very important) ising and potts models

def let $G = (V,E)$ be a finite graph. let $p \in [0,1]$ and $q > 0$, and let $\Omega = \{0,1\}^E$.

define $\phi_{G,p,q} : \Omega \to [0,1]$

as $$\phi_{G,p,q}(\omega) = \frac{1}{Z_{G,p,q}} p^{o(\omega)} (1-p)^{c(\omega)} \cdot q^{k(\omega)}$$

where $k(\omega)$ is the number of clusters of ω.

(recall every vertex is in a cluster, and clusters can be just a single vertex).

where $Z_{G,p,q} = \sum_{\omega \in \Omega} p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}$

rmk increasing q means the measure favours more clusters

$\Rightarrow$ heuristically, clusters are smaller

vice-versa for q smaller.

rmk we no longer have $\omega_e, \omega_{e'}$ independent!

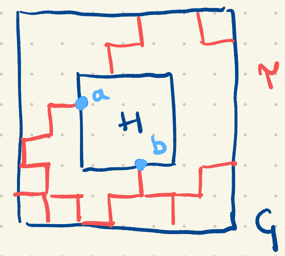
boundary conditions

- let $H \subset G$ a subgraph. we want to make sense of

$$\phi_{G,p,q}\left[\omega|_H \;\middle|\; \omega|_{G \setminus H} = \tau \right]$$

where $\tau \in \{0,1\}^{E(G) \setminus E(H)}$.

eg



notice a & b are connected by γ

$\Rightarrow$ the cluster in H touching a

=

the cluster in H touching b

this changes the definition of $k(\omega)$, ie $k(\omega)$ depends on the boundary conditions.

def (boundary conditions)

let ξ be any partition of $\partial G \subset V$.
(ie a set of disjoint subsets $A_i \subset \partial G$ st. $\bigcup_i A_i = \partial G$)

then define $x, y \in \partial G$ to be in the same cluster if $x, y \in A_i$ for some A_i.

now define $k^{\xi}(\omega) =$ # clusters with above defn added.

then $\phi^{\xi}_{G,p,q}(\omega) = \frac{1}{Z^{\xi}_{G,p,q}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)}$

$\boxed{\text{eg}}$ $\xi = \{\{x_1\}, \{x_2\}, \dots\}$ = singletons

this gives our original definition back

we call this <u>free</u> boundary conditions

& write $\phi^0_{G,p,q}$

$\boxed{\text{eg}}$ $\xi = \{\partial G\}$ ie. all elts of ∂G connected

this is "the opposite" of free

— we call this <u>wired</u> boundary conditions

& write $\phi^1_{G,p,q}$.

<u>thm</u> for A increasing, $p < p'$, $1 \le q < q'$,

$$\phi^\xi_{G,p,q}[A] \le \phi^\xi_{G,p',q}[A]$$

$$\phi^\xi_{G,p,q}[A] \ge \phi^\xi_{G,p,q'}[A]$$

say $\xi \leq \xi'$ if $\forall A_i \in \xi$ $\exists B_j \in \xi'$ s.t. $A_i \subset B_j$.

<u>note</u> $\forall \xi$, $0 \leq \xi \leq 1$
free wired

<u>thm</u> (comparison of boundary conditions) $\forall A$ increasing, $\xi \leq \xi'$

$$\phi^{\xi}_{G,p,q}[A] \leq \phi^{\xi'}_{G,p,q}[A]$$

<u>thm</u> (FKG) $\forall A, B$ increasing,

$$\phi^{\xi}_{G,p,q}[A \cap B] \geq \phi^{\xi}_{G,p,q}[A]\, \phi^{\xi}_{G,p,q}[A].$$

<u>thm</u> (domain markov property)

if $\tau \in \{0,1\}^{E(G) \setminus E(H)}$, let $\xi(\tau)$ be the partition of ∂H given by x, y in same set $\iff$ $x \overset{\tau}{\longleftrightarrow} y$.

then $\phi^{\xi}_{G,p,q}\left[\omega|_H = \sigma \;\middle|\; \omega|_{G\setminus H} = \tau \right]$

$$= \phi^{\xi(\tau)}_{H,p,q}\left[\sigma\right]$$

this is the best we can do to replace independence.

- one can go on to show $\exists$ measures on $\mathbb{Z}^d$
- for $q \geq 2$, $q \in \mathbb{N}$, FK-percolation is closely connected with a very important model : the potts model (when $q=2$ this is the ising model).