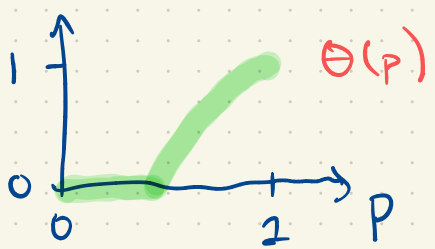
$\boxed{2}$ existence of phase transition

def $\theta : [0,1] \to [0,1]$ as
$$\theta(p) = \mathbb{P}_p[0 \leftrightarrow \infty],$$
where $0 = (0,\dots,0) \in \mathbb{Z}^d$ is the origin.

def the critical point is defined as
$$p_c = p_c(d) = \inf\{p \in [0,1] : \theta(p) > 0\}$$

also **def** $\psi : [0,1] \to [0,1]$ as
$$\psi(p) = \mathbb{P}_p[\underbrace{\text{percolation}}_{(\exists\, \infty \text{ cluster})}]$$

- expected behaviour : $\theta(0) = 0$, $\theta(1) = 1$, monotonic



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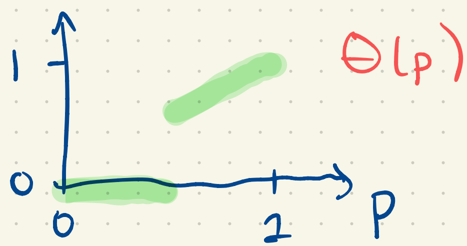
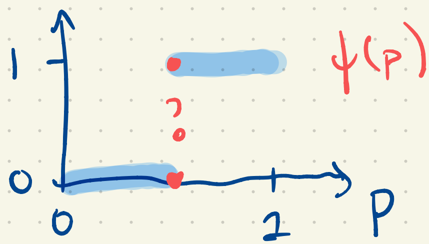




- we will prove monotonicity in the next chapter.

ex show $p_c(1) = 1$.

thm (non-trivial phase transition for $d \geq 2$)

for $d \geq 2$, $0 < p_c(d) < 1$.

proof $\boxed{p_c > 0}$

- let Υ_L be the set of paths γ length L starting at origin. for all L,

$$\theta(p) = \mathbb{P}_p[0 \leftrightarrow \infty] \leq \mathbb{P}_p[\exists \gamma \in \Upsilon_L : \gamma \text{ open}]$$

$$= \mathbb{P}_p\Big[\bigcup_{\gamma \in \Upsilon_L} \{\gamma \text{ open}\}\Big]$$

$$\underset{\text{union bound}}{\leq} \sum_{\gamma \in \Upsilon_L} \mathbb{P}_p[\gamma \text{ open}]$$

$$= \sum_{\gamma \in \Upsilon_L} p^L$$

$$= p^L \cdot |\Upsilon_L|$$

$$\leq p^L \cdot 2d \cdot (2d-1)^{L-1}$$

now if $p < \frac{1}{2d-1}$ then this is

$$p \cdot 2d \cdot \lambda^{L-1}, \quad \lambda < 1.$$

- this is true $\forall L$, so $\theta(p) = 0$ for $p < \frac{1}{2d-1}$, so $p_c(d) \geq \frac{1}{2d-1}$.

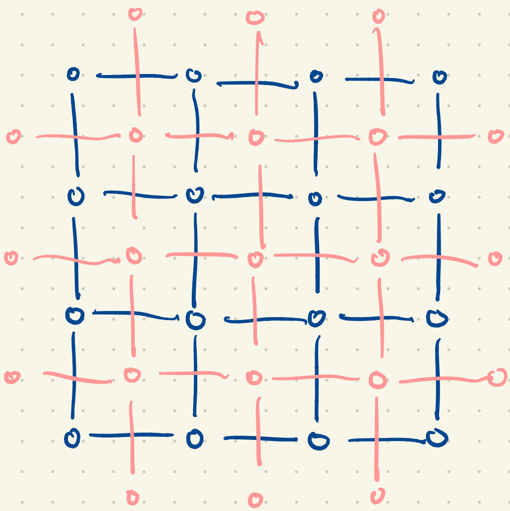
$\boxed{p_c < 1}$ we need to prove $\theta(p) > 0$ for p close to 1.

- reduce to $\mathbb{Z}^2$: observe that percolation on $\mathbb{Z}^d$, $d \geq 2$ contains a copy of percolation on $\mathbb{Z}^2$ (just restrict to the configuration on $\mathbb{Z}^2 \subset \mathbb{Z}^d$).

therefore if $\theta_2(p) > 0$, then $\theta_d(p) > 0$

$$\Rightarrow p_c(d) \leq p_c(2) \qquad \forall d \geq 2$$

- prove for $\mathbb{Z}^2$: planar duality



- consider the dual graph of $\mathbb{Z}^2$:

$$(\mathbb{Z}^2)^* = (F, E^*)$$

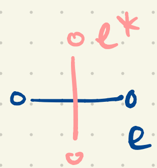
$F = \{ \text{faces of } \mathbb{Z}^2 \}$

$E^* = \{ \{f_1, f_2\} : f_1, f_2 \text{ share an edge} \}$

- for each configuration $\omega \in \{0,1\}^E$, there is a dual configuration $\omega^* \in \{0,1\}^{E^*}$

as: $\omega^*_{e^*} = 1 - \omega_e$

where e^* is the dual edge to e :



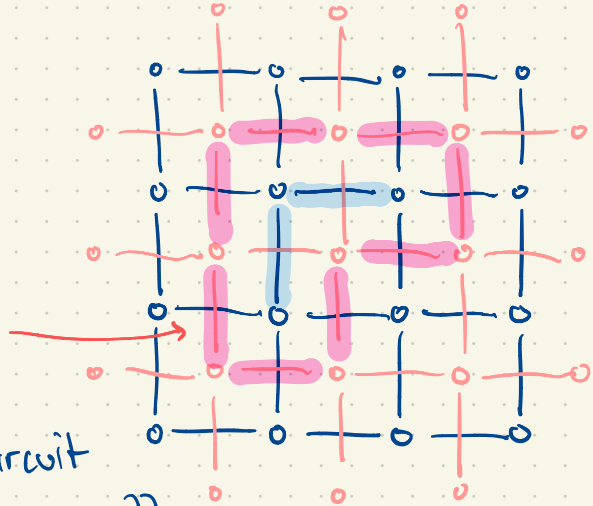
- $(\mathbb{Z}^2)^*$ is a copy of $\mathbb{Z}^2$, and ω^* is distributed as Bernoulli bond percolation with parameter $p^* = 1-p$.

so :

- $1 - \Theta_2(p)$

$= \mathbb{P}_p[\text{cluster of } 0 \text{ is finite}]$

$= \mathbb{P}_p\left[\omega^* \text{ has an open circuit around } 0\right]$



$$= \mathbb{P}_p\left[\bigcup_{k \geq 0} \left\{\omega^* \text{ has open circuit round } 0, \text{ passes thru } \left(k+\tfrac{1}{2}, \tfrac{1}{2}\right)\right\}\right]$$

$$\underset{\text{bound}}{\overset{\text{union}}{\leq}} \sum_{k \geq 0} \mathbb{P}\left[\omega^* \text{ has open circuit round } 0, \text{ passes thru } \left(k+\tfrac{1}{2}, \tfrac{1}{2}\right)\right]$$

$$\underset{\text{bound}}{\overset{\text{union}}{\leq}} \sum_{k \geq 0} \sum_{\substack{\gamma \text{ circuit in } (\mathbb{Z}^2)^* \\ \text{round } 0, \text{ thru } (k+\frac{1}{2}, \frac{1}{2})}} \mathbb{P}_p[\gamma \text{ open in } \omega^*]$$

$$\leq \sum_{k \geq 0} \sum_{l \geq 2k+4} \sum_{\substack{\gamma \text{ — '' —} \\ \text{and } |\gamma| = l}} (1-p)^l$$

$$\leq \sum_{k \geq 0} \sum_{l \geq 2k+4} (4(1-p))^l$$

$$= \sum_{k \geq 0} (4(1-p))^{2k+4} \sum_{l \geq 0} (4(1-p))^l$$

$\sum_{l \geq 0} (4(1-p))^l = c(p)$, finite and decreasing for $p > 3/4$, and $\to 1$ as $p \to 1$

$$= c(p)\,(4(1-p))^4 \cdot \sum_{k \geq 0} \left[(4(1-p))^2\right]^k$$

$\sum_{k \geq 0} (4(1-p))^2 = c'(p)$, finite and decreasing for $p > 3/4$. and $\to 1$ as $p \to 1$

$$= c(p)\,c'(p)\,(4(1-p))^4 \to 0 \text{ as } p \to 1$$

so $\exists\, p_0 \in (0,1)$ st.

the above is < 1 for all $p > p_0$.

hence for $p > p_0$, $1 - \Theta(p) < 1$

$\iff$

$\Theta(p) > 0$.

hence $p_c \leq p_0 < 1$. ∎

extras how general can we make the work above?

thm (above) $p_c(\mathbb{Z}) = 1$ (exercise above),
and $p_c(\mathbb{Z}^d) \in (0,1)$.

def a graph G is **transitive** if $\forall x, y \in G$, $\exists$ graph automorphism ϕ st. $\phi(x) = y$. ϕ is a **graph automorphism** if $\phi : G \to G$ st.

$$\phi(x) \sim \phi(y) \iff x \sim y$$

thm (duminil-copin, goswami, raofi, severo, yadin 2020)
(easo, hutchcroft 2023)

let G be an infinite, transitive graph. we say G is one-dimensional if $|B_n| = O(n)$ as $n \to \infty$. we have that:

$$p_c(G) \in (0,1) \iff G \text{ not one-dimensional}$$