3 monotonicity

recall : def $\theta : [0,1] \to [0,1]$ as

$$\theta(p) = \mathbb{P}_p[0 \leftrightarrow \infty]$$

def $\psi : [0,1] \to [0,1]$ as

$$\psi(p) = \mathbb{P}_p[\text{percolation}]$$
($\exists\, \infty$ cluster)

prop θ, ψ non-decreasing in p.

proof • we will use a coupling between $\mathbb{P}_p$ and $\mathbb{P}_{p'}$, $p' > p$.

rmk a coupling is a "larger" probability space, with more information, such that the two measures μ_1, μ_2 are "contained within" this space

(1) measurable functions :

def let $(\Omega_1, \mathbb{F}_1)$, $(\Omega_2, \mathbb{F}_2)$ measurable spaces (sets & σ-algebras on them). a function $f : \Omega_1 \to \Omega_2$ is measurable if

$$\forall A \in \mathbb{F}_2, \quad f^{-1}(A) \in \mathbb{F}_1$$

def we write $\mathcal{B} := \sigma(\text{open intervals in } \mathbb{R})$. in the case $(\Omega_2, \mathbb{F}_2) = (\mathbb{R}, \mathcal{B})$, we simply say f is a (Borel) **measurable function on Ω_1**.

def / lem let $(\Omega_1, \mathbb{F}_1, \mu_1)$ measure space, $(\Omega_2, \mathbb{F}_2)$ a measurable space; let $f : \Omega_1 \to \Omega_2$ measurable. then $\mu_2 := \mu_1 \circ f^{-1}$ as

$$\mu_2(A) = \mu(f^{-1}(A))$$

is a measure (the **image measure**) on Ω_2.

② random variables

def let $(\Omega, \mathbb{F}, \mathbb{P})$ a probability space. let $(\Omega_2, \mathbb{F}_2)$ a measurable space. a **random variable** X is

a measurable function $X : \Omega \to \Omega_2$.

if we don't specify $(\Omega_2, \mathbb{F}_2)$, we assume it is $(\mathbb{R}, \mathcal{B})$.

def $\mu_X : \mathbb{P} \circ X^{-1}$ is a measure on $\mathbb{R}$ (or Ω_2), called the **law** or **distribution** of X. we write $X \sim \mu_X$.

- different RVs on different probability spaces can have the same law.

$$\begin{tikzcd}
(\Omega, \mathbb{F}) \arrow[r, "X"] \arrow[d, "\mathbb{P}"'] & (\Omega_2, \mathbb{F}_2) \arrow[dl, bend right, "\mathbb{P}\circ X^{-1}"'] \\
\mathbb{R} &
\end{tikzcd}$$

law of X

def $(X_i : i \in I)$ RVs on $(\Omega, \mathbb{F}, \mathbb{P})$ are **independent** if the σ-algebras $\sigma(X^{-1}(\mathbb{F}_2))$ are independent.

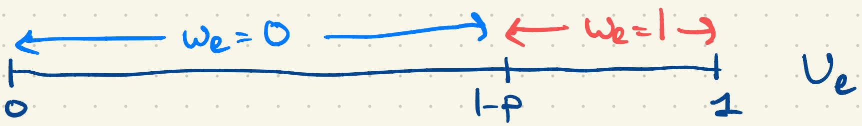
def say μ_1, μ_2 are measures on Ω_1, Ω_2, respectively. a **coupling** is a prob. space $(\Omega^*, \mathbb{F}^*, \mathbb{P}^*)$ and a pair of RVs X_1, X_2 st. $X_i : \Omega^* \to \Omega_i$ st. the law of X_i is μ_i : $\mathbb{P}^* \circ X_i^{-1} = \mu_i$

$$\begin{tikzcd}
& (\Omega^*, \mathbb{F}^*) \arrow[dl, "X_1"'] \arrow[dr, "X_2"] \arrow[dd, "\mathbb{P}^*"] \arrow[dd, bend right=40, "\mathbb{P}^*\circ X_2^{-1}"'] \arrow[dd, bend left=40, "\mathbb{P}^*\circ X_1^{-1}"] & \\
(\Omega_1, \mathbb{F}_1) \arrow[dr, "\mu_1"'] \arrow[r, phantom, "="] & & (\Omega_2, \mathbb{F}_2) \arrow[dl, "\mu_2"] \arrow[l, phantom, "="] \\
& \mathbb{R} &
\end{tikzcd}$$

construct our coupling:

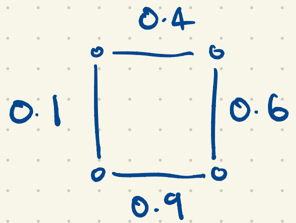
- on every edge e put a uniform RV on $[0,1]$, called U_e, independent of one another.
  $(\Omega^* = [0,1]^E,\ \mathbb{F}^* = \text{Borel cylinders},\ \mathbb{P}^* = \text{unif}^{\otimes E})$ $\boxed{*}$
- $\{U_e\}_{e \in E}$: $U_e \sim \text{Unif}[0,1]$
  U_e, U_f indep $\forall e \neq f$
- let $p \in [0,1]$. define $\omega \in \{0,1\}^E$ as
  - $\omega_e = 1$ if $U_e \geq 1-p$
  - $\omega_e = 0$ if $U_e < 1-p$.

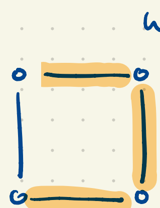
eg



eg

$p = 0.7$





claim $\omega \sim \mathbb{P}_p$

proof
- $\mathbb{P}[\omega_e = 1] = \mathbb{P}[U_e \geq 1-p] = p$
- ω_e is a function of U_e
- $\{U_e\}_{e\in E}$ indep, so $\{\omega_e\}_{e \in E}$ indep ∎

proof of prop

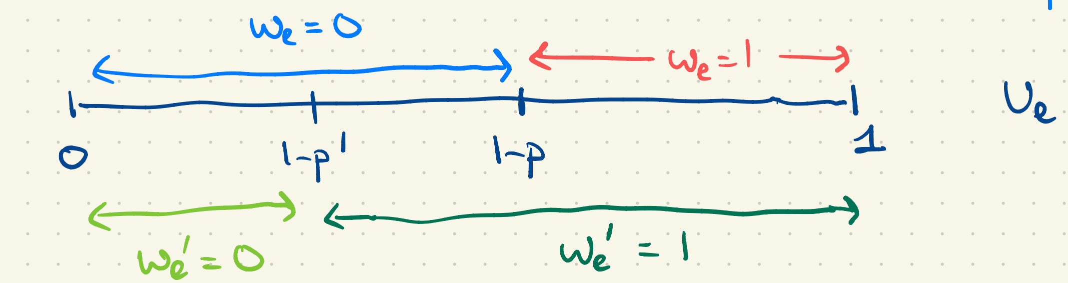
- now take p' st. $p \leq p' \leq 1$. define w' as
  - $w'_e = 1$ if $U_e \geq 1-p'$
  - $w'_e = 0$ if $U_e < 1-p'$

as before, $w' \sim \mathbb{P}_{p'}$

- crucially, we used the same U_e, so

$$\mathbb{P}^*[w_e \leq w'_e] = 1 \quad \forall e \in E$$

(indeed, $\mathbb{P}^*[w_e > w_e'] = \mathbb{P}^*[w_e = 1, w_e' = 0] = \mathbb{P}^*[U_e \geq 1-p, U_e < 1-p'] = 0$)



- we can write the above more concisely.

def for $w, w' \in \{0,1\}^E$, we say $w \leq w'$ if $w_e \leq w'_e \ \forall e \in E$.

- so by the above, we have $\mathbb{P}^*[w \leq w'] = 1$.

- notice that for $\omega, \omega' \in \{0,1\}^E$, if $0 \leftrightarrow \infty$ in ω, and $\omega \le \omega'$, then $0 \leftrightarrow \infty$ in ω'. $\boxed{*}$

- hence $\Theta(p) = \mathbb{P}^*[0 \overset{\omega}{\leftrightarrow} \infty] = \mathbb{P}^*[0 \overset{\omega}{\leftrightarrow} \infty, \omega \le \omega']$

$$\le \mathbb{P}^*[0 \overset{\omega'}{\leftrightarrow} \infty] = \Theta(p')$$

$\blacksquare$

(similar for $\psi(p)$).

- let's examine the proof above. what property of the events $\{0 \overset{\omega}{\leftrightarrow} \infty\}$ and $\{\text{percolation}\}$ makes the proof work? it's property $\boxed{*}$

def an event A is **increasing** if the following holds:

$$\omega \in A \text{ and } \omega' \geq \omega \Rightarrow \omega' \in A$$

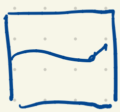
forall $\omega, \omega' \in \{0,1\}^E$.

rmk A, B increasing $\Rightarrow$ $A \cup B$, $A \cap B$ increasing

rmk A **decreasing** if A^c increasing

eg $\{x \leftrightarrow y\}$ increasing, $\{x \not\leftrightarrow y\}$ decreasing

$\{$ the cluster at x has size $\geq 5 \}$ increasing

$\{$  open crossing of box $\}$ increasing

prop there exists (3rd) probability space $(\Omega^*, \mathbb{F}^*, \mathbb{P}^*)$ and random variables $\omega^{(p)} : \Omega^* \to \{0,1\}^E$ st. $\forall p \in [0,1]$, the law of $\omega^{(p)}$ under $\mathbb{P}^*$ is $\mathbb{P}_p$, and

$$\mathbb{P}^*\left[\omega^{(p)} \leq \omega^{(p')}\right] = 1 \qquad \forall p, p'.$$

cor $\boxed{a}$ let $A \in \mathcal{F}$ be increasing. then

$\mathbb{P}_p[A]$ non-decreasing as fn of p.

$\boxed{b}$ let $f : \{0,1\}^E \to \mathbb{R}$ measurable and increasing

$(\omega \leq \omega' \Rightarrow f(\omega) \leq f(\omega'))$ $\boxed{*}$

≥ 0 or bounded

then $\mathbb{E}_p[f] = \int f \, d\mathbb{P}_p$ nondecreasing in p

proof of **prop** and **cor**

- run proof of prop above to give $\mathbb{P}[\omega_p \leq \omega_{p'}] = 1$.
- $\boxed{b} \Rightarrow \boxed{a}$, indeed, let $f = \mathbb{1}_A$.
- $\mathbb{E}_p[f] = \mathbb{E}[f(\omega_p)] \leq \mathbb{E}[f(\omega_{p'})] = \mathbb{E}_{p'}[f]$

■

def recall a coupling between measures μ_1 on Ω_1 and μ_2 on Ω_2 is prob. space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and a pair of RVs X_1, X_2 st. $X_i : \Omega^* \to \Omega_i$ st. the law of X_i is μ_i : $\mathbb{P}^* \circ X_i^{-1} = \mu_i$

def let μ_1, μ_2 be measures on <u>same</u> space Ω, Ω <u>ordered</u>. a coupling $(\Omega^*, \mathbb{F}^*, \mathbb{P}^*)$, X_1, X_2 of μ_1, μ_2 is a **monotone coupling** if

$$\mathbb{P}^*[X \le X'] = 1$$

we say μ' **stochastically dominates** μ; $\mu \overset{st}{\le} \mu'$, if $\exists$ such a coupling.

<u>rmk</u> monotone couplings are extremely useful.

prop $\mathbb{P}_p \overset{st}{\le} \mathbb{P}_{p'}$ for all $p' \geq p$.

<u>proof</u> done above ■

eg let μ = bernoulli(p), μ' = bernoulli(p'), $p \le p'$.