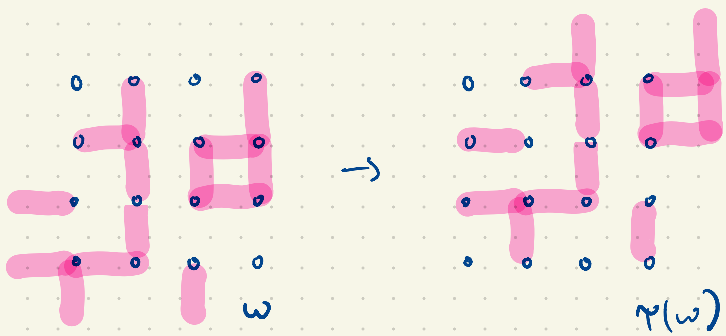
$\boxed{4}$ ergodicity (and translation-invariance and mixing)

rmk the graph $\mathbb{Z}^d$ has symmetries: rotations, reflections, translations. how does $\mathbb{P}_p$ relate to these symmetries?

- let $\gamma : \mathbb{Z}^d \to \mathbb{Z}^d$ be a graph isomorphism, that is, $x \sim y \iff \gamma(x) \sim \gamma(y) \quad \forall x, y \in \mathbb{Z}^d$.
- for $\omega \in \{0,1\}^E$, let $\gamma(\omega) \in \{0,1\}^E$ as

$$\gamma(\omega)(xy) = \omega(\gamma^{-1}(x)\,\gamma^{-1}(y)).$$

eg if γ is translation by $(1,1)$



- for $z \in \mathbb{Z}^d$, denote translation by z by γ_z
- if A an event, let $\gamma(A) = \{\omega : \gamma^{-1}(\omega) \in A\}$
- if $A = \{x \leftrightarrow y\}$, $\gamma_z(A) = \{x+z \leftrightarrow y+z\}$
- all translations, reflections, rotations are graph isomorphisms. are there others?

def • a **measure** μ on $\{0,1\}^E$ is **invariant to translations** if, $\forall x \in \mathbb{Z}^d$, $\forall$ events $A \in \mathbb{F}$,

$$\mu(\tau_x A) = \mu(A).$$

prop our measure $\mathbb{P}_p$ is invariant to translations

proof $\boxed{a}$ show for cylinder events.

let A be a cylinder event, dependent on finite set of edges E. then

$$\mathbb{P}_p[\tau_x A] = \sum_{\substack{\omega \in \{0,1\}^{\tau_x E} \\ \omega \in \tau_x A}} p^{o(\omega)} (1-p)^{c(\omega)}$$

$$= \sum_{\substack{\omega \in \{0,1\}^E \\ \omega \in A}} p^{o(\omega)} (1-p)^{c(\omega)} = \mathbb{P}_p[A].$$

$\boxed{b}$ extend to whole σ-alg. let $\tilde{\mathbb{P}}_p[A] = \mathbb{P}_p[\tau_x A]$ $\forall$ events A. then $\tilde{\mathbb{P}}_p = \mathbb{P}_p$ on cylinder events, so is a premeasure. now $\tilde{\mathbb{P}}_p = \mathbb{P}_p$ extends to a unique measure on $\mathbb{F}$, which must be $\mathbb{P}_p$, by (uniqueness in) caratheodory's extension theorem.

def an event $A \in \mathbb{F}$ is invariant to translations if

$$\tau_z A = A \qquad \forall z \in \mathbb{Z}^d$$

eg
- $\{ \exists$ a cluster of size $n \}$
- $\{ \exists \infty$ cluster $\}$
- $\{ 0 \leftrightarrow \infty \}$ <u>not</u> trans-inv.

def a measure μ on $\{0,1\}^E$ is ergodic if $\forall$ events $A \in \mathbb{F}$ invariant to translations,

$$\mu(A) \in \{0,1\}$$

def a measure μ on $\{0,1\}^E$ is mixing if $\forall$ events $A, B \in \mathbb{F}$,

$$\lim_{|z| \to \infty} \mu[A \cap \tau_z B] = \mu[A]\mu[B]$$

<u>rmk</u> "translate B far away $\Rightarrow$ A & B become independent"

prop (mixing) $\mathbb{P}_p$ is mixing.

proof fix events A, B. let $\varepsilon > 0$. by cor to carathéodory, $\exists$ events $A_\varepsilon, B_\varepsilon$ dependent on finitely many edges E, E', respectively, such that

$$\mathbb{P}_p[A \Delta A_\varepsilon] \leq \varepsilon \quad \text{and} \quad \mathbb{P}_p[B \Delta B_\varepsilon] \leq \varepsilon .$$

now for $|z|$ large enough, $\tau_z E'$, E are disjoint, so $\tau_z B_\varepsilon$, A_ε are independent, so

$\boxed{*}$ use of independence

$$\begin{aligned}\mathbb{P}_p[A_\varepsilon \cap \tau_z B_\varepsilon] &= \mathbb{P}_p[A_\varepsilon]\,\mathbb{P}_p[\tau_z B_\varepsilon] \\ &\underset{\text{translation-invariance}}{=} \mathbb{P}_p[A_\varepsilon]\,\mathbb{P}_p[B_\varepsilon].\end{aligned}$$

now, for $|z|$ large enough,

$$\begin{aligned}\mathbb{P}_p[A \cap \tau_z B] &\leq \mathbb{P}_p[A_\varepsilon \cap \tau_z B_\varepsilon] + 2\varepsilon \\ &= \mathbb{P}_p[A_\varepsilon]\,\mathbb{P}[B_\varepsilon] + 2\varepsilon \\ &\leq \mathbb{P}_p[A]\,\mathbb{P}[B] + 4\varepsilon\end{aligned}$$

& similarly, $\mathbb{P}_p[A \cap \tau_z B] \geq \mathbb{P}_p[A]\,\mathbb{P}_p[B] - 4\varepsilon$

$\blacksquare$

prop $\mathbb{P}_p$ is ergodic.

proof let A be a translation-invariant event. then

$$\lim_{|z|\to\infty} \mathbb{P}_p[A \cap \tau_z A] = \mathbb{P}_p[A]\,\mathbb{P}[A] = \mathbb{P}[A]^2$$

and

$$\text{LHS} = \lim_{|z|\to\infty} \mathbb{P}_p[A \cap A] = \mathbb{P}[A]$$

$\blacksquare$

eg

- $\mathbb{P}_p[0 \leftrightarrow \infty,\ z \leftrightarrow \infty] \xrightarrow[|z|\to\infty]{} \mathbb{P}_p[0 \leftrightarrow \infty]^2 = \theta(p)^2.$

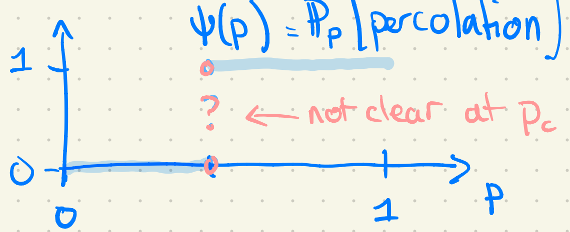
- let $N(\omega) = \#$ infinite clusters in ω. then $\forall k \in \mathbb{N}$,

$$\mathbb{P}_p[N = k] \in \{0,1\}.$$

in particular, $\mathbb{P}_p[N > 0] = \mathbb{P}_p[\text{percolation}] \in \{0,1\}.$

- and further, $\exists k \in \mathbb{N} \cup \{\infty\}$ st. $\mathbb{P}_p[N = k] = 1.$
  (k may depend on p).

rmk now we have



<u>questions</u>: what is $\psi(p_c)$?

when $\exists$ ∞ cluster, how many exist ?