$\boxed{5}$ unique infinite cluster

thm (aizenman, kesten, newman 87). let $p \in [0,1]$. then either

$$\mathbb{P}_p[N=0]=1 \quad \text{or} \quad \mathbb{P}_p[N=1]=1.$$

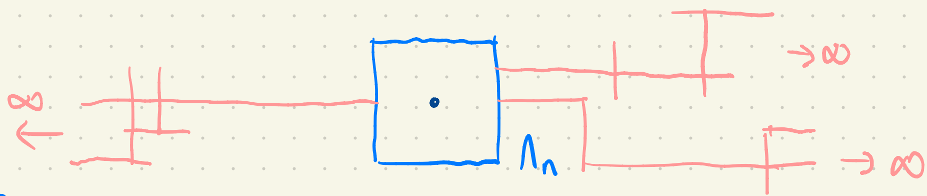
proof: we follow the proof of (burton, keane 89).

- let $p \in [0,1]$. recall that by ergodicity, $\exists\, k=k(p) \in \mathbb{N} \cup \{\infty\}$ st. $\mathbb{P}_p[N=k]=1$.

part $\boxed{1}$: $k \in \{0,1,\infty\}$.

assume, for contradiction, that $1<k<\infty$. so $\mathbb{P}[N=k]=1$, $\mathbb{P}[N=1]=0$.

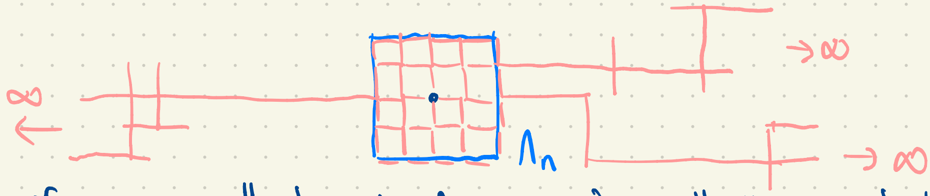
$$I_n = \{ \text{all the } \infty \text{ clusters intersect } \Lambda_n \}$$



this is monotone convergence (see chapter 5.5)

- $I_n \subset I_{n+1}$, and since we assumed $\mathbb{P}_p[N=k]=1$, we have $\mathbb{P}_p[I_n] \nearrow \mathbb{P}_p[N=k]=1$ ← this doesn't hold if $k=\infty$, as then $\mathbb{P}_p[I_n]=0\ \forall n$.

$\Rightarrow \exists\, n_0 \in \mathbb{N}$ st. $\forall n \geq n_0$, $\mathbb{P}_p[I_n] > 0$.



- if we open all edges in Λ_n, we join all the ∞ clusters!

- now $\mathbb{P}_p[N=1]$

$$\geq \mathbb{P}_p[I_n \cap \{\text{all edges open in } \Lambda_n\}]$$

here we use that $I_n = \{\text{all } k\ \infty \text{ clusters intersect } \Lambda_n\}$ is dependent only on edges outside Λ_n, so is indep. of $\{\text{all edges open in } \Lambda_n\}$.

$$\boxed{=} \mathbb{P}_p[I_n] \cdot p^{|\Lambda_n|}$$

$$> 0 \quad , \quad \text{contradiction.}$$

rmk more generally, the above property works for any measure μ on $\{0,1\}^E$ st:

$$\forall n > 0, \ \exists c(n) > 0 \quad \text{st. } \forall S \subset E, \ |E| \leq n,$$

$\forall$ events $A \subset \{0,1\}^{E \setminus S}$ st. $\mu(A) > 0$, $\sigma \in \{0,1\}^S$,

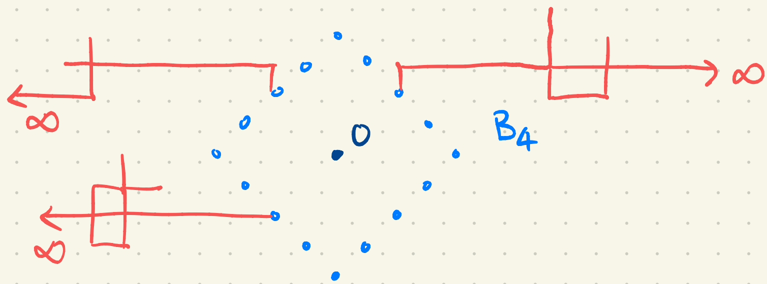
$$\mu[\omega|_S = \sigma \mid A] \geq c.$$

this is called the finite energy property.

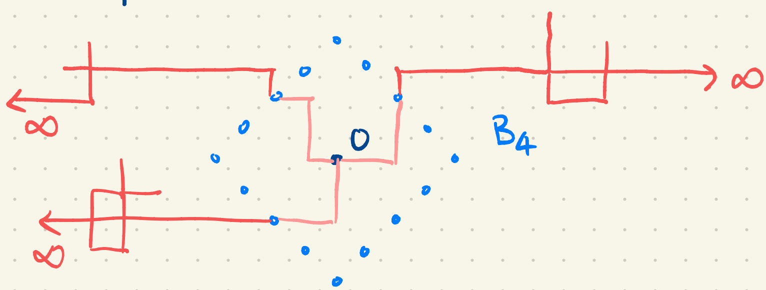
- one should think of this as : "we can change the configuration in a box size n for a finite cost (only dependent on n)"

<u>part</u> $\boxed{2}$: $k \neq \infty$.

- assume $k \geq 3$ for contradiction.
  let $n > 0$ large such that (monotone convergence ; see chapter $\boxed{5.5}$)
  $$\mathbb{P}_p\Big[\ \exists\ 3\ \infty \text{ clusters } \underset{(\text{outside } B_n)}{\text{intersecting}} \text{ box } B_n\Big] > 0$$
  where B_n = ball around 0 of graph-distance radius n:



- using finite energy, we can let these $3\ \infty$ clusters join exactly at 0, at finite cost.



- let's be precise. let T_0 be the event that there are 3 ∞ clusters which are connected in ω but not in $\omega|_{\mathbb{Z}^d \setminus \{0\}}$. let $T_z = \tau_z T_0$. we say there is a trifurcation at z if T_z happens.

- $\mathbb{P}_p[T_0] \geq \mathbb{P}[\geq 3\ \infty$ clusters intersecting B_n, these 3 clusters join in B_n only at $0]$

  $\geq \mathbb{P}[\geq 3\ \infty$ clusters intersecting $B_n] \cdot c(|B_n|)$

  > 0

- hence $\mathbb{E}[\#$ trifurcation points in $B_n]$

$$= \sum_{x \in B_n} \mathbb{P}[T_x]$$

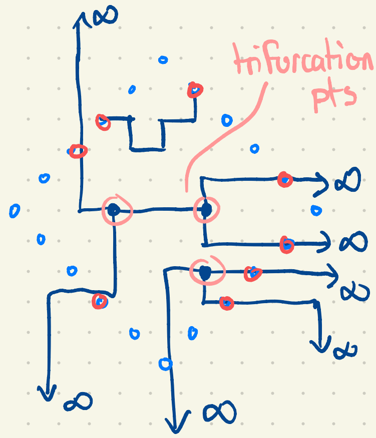
$$= |B_n| \, \mathbb{P}[T_0].$$

this is one half of our contradiction. for the other half, we will show that $T_n = \#$ trifurcations in B_n is $\leq |\partial B_n|$ deterministically, so $\mathbb{E}[T_n] \leq |\partial B_n|$. then

$$\frac{|\partial B_n|}{|B_n|} \to 0$$ gives the contradiction.

- let $\omega \in \{0,1\}^E$ such that $\omega|_{B_n}$ is a forest (has no cycles) with all its leaves (degree 1 vertices) in ∂B_n. then

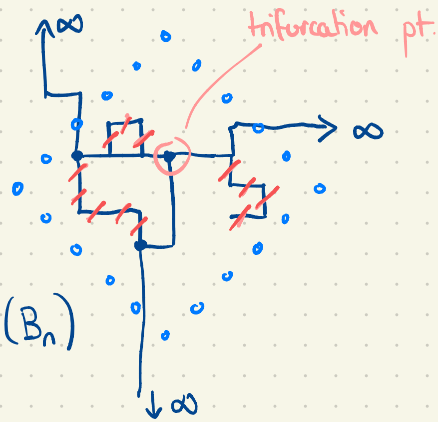
$$T_n(\omega) \leq \#\text{ leaves of } \omega|_{B_n}$$
$$\leq |\partial B_n|.$$



- now we reduce to the case above. take any $\omega \in \{0,1\}^E$ and remove its cycles, then remove its leaves not in ∂B_n.

<u>claim</u> this cannot remove any trifurcations, so

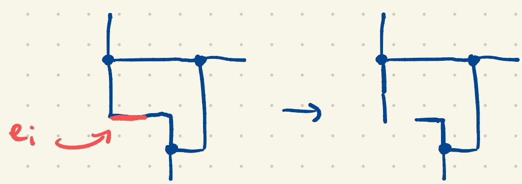
$$T_n(\omega) \leq T_n(\omega_{\text{modified}}) \leq |\partial B_n|$$



- let us define this process rigorously, & prove the claim.

let $F_0 := \{e_1, \dots, e_r\}$ = edges in $E(B_n)$ open in ω.

ⓐ $\forall\, 1 \leq i \leq r$, if e_i is on a cycle of edges in F_{i-1}, set $F_i = F_{i-1} \setminus \{e_i\}$; otherwise set $F_i = F_{i-1}$.

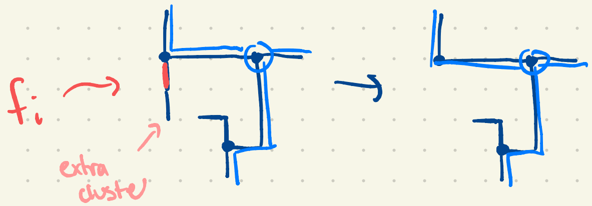


doesn't remove trifurcations, as $x \leftrightarrow y$ in ω is preserved $\forall\, x, y$. (can add trifurcations).

then $\tilde{\tilde{F}}_0 := F_r = \{f_1, \dots, f_s\}$ is a forest.

(b) $\forall 1 \le j \le s$, if $\tilde{\tilde{F}}_{j-1} \setminus \{f_j\}$ has a cluster not intersecting ∂B_n (can be one vertex), set $\tilde{\tilde{F}}_{j-1} \setminus (\{f_j\} \cup \text{that cluster})$

otherwise, set $\tilde{\tilde{F}}_j = \tilde{\tilde{F}}_{j-1}$.



if x is a trifurcation point, $\exists \geq 3$ paths $x \leftrightarrow B_n$. edges on these paths can't be removed.

at the end, $\tilde{\tilde{F}}_s$ is a forest whose leaves are in ∂B_n

$\blacksquare$

<u>rmk</u> all we used in part 2 was ergodicity and

$$\lim_{n \to \infty} \frac{|\partial B_n|}{|B_n|} = 0.$$

def a (countably) infinite, locally finite ($\deg(x) < \infty$ $\forall x \in G$) transitive graph is called **amenable** if

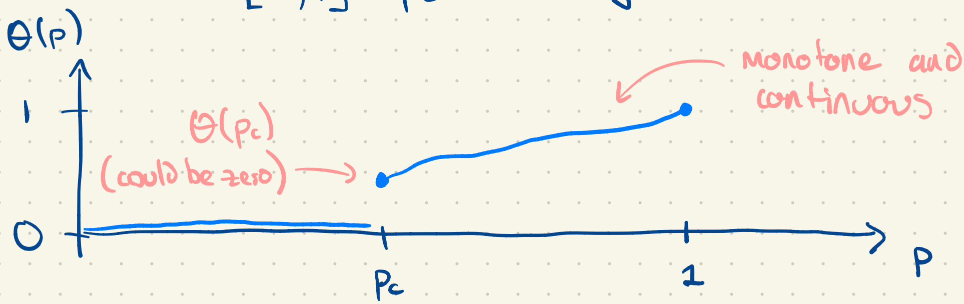
$$\inf_{H \subset G} \frac{|\partial H|}{|H|} = 0$$

(G is transitive if $\forall x, y \in G$, $\exists$ graph automorphism

ϕ st. $\phi(x) = y$. ϕ is a graph automorphism if $\phi: G \to G$ st. $\phi(x) \sim \phi(y) \Leftrightarrow x \sim y$).

$$=$$

prop the map $\theta(p) = \mathbb{P}_p[0 \leftrightarrow \infty]$ is continuous on $[0,1] \setminus p_c$ and right-continuous at p_c.

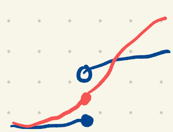


proof (a) θ is right-continuous.

we know that $\theta_k(p) = \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_k]$ is polynomial in p, so continuous, & is nondecreasing in p. moreover, $\theta_k(p)$ is non-increasing in k. so $\theta = \lim_{k\to\infty} \theta_k$ is a "decreasing limit of $_{\text{cts}}$ increasing functions".

this gives right-continuity.

we want:



$$\lim_{x_n \downarrow x} \theta(x_n) = \theta(x)$$

let $y^+ = \text{LHS} = \lim_{x_n \downarrow x} \lim_{k \to \infty} \theta_k(x_n)$,

$$y^- = RHS = \lim_{k\to\infty} \Theta_k(x).$$

$\circledast$

we know that $\forall n \in \mathbb{N}$, $\forall k \in \mathbb{N}$, $\Theta_k(x_n) \geq y^+$

if $y^+ > y^-$ then $\exists k_0 \in \mathbb{N}$ st. $k > k_0$

$$\Rightarrow \Theta_k(x) < (y^+ + y^-)/2$$

now as Θ_k cts, $\exists n_0 \in \mathbb{N}$ st. $n \geq n_0$

$$\Rightarrow \quad \Theta_k(x_n) < \frac{3y^+ + y^-}{4} < y^+$$

contradicts $\circledast$.

(b) Θ left-continuous on $[0,1] \setminus p_c$.

well, $[0, p_c)$ is trivial. fix $p_0 > p_c$. we want

$$\lim_{p \nearrow p_0} \Theta(p) = \Theta(p_0).$$

recall we coupled the measures $\mathbb{P}_p$, $p \in [0,1]$ together; let $\mathbb{P}^*$ be the joint measure.

$$\lim_{p\nearrow p_0} \Theta(p) = \lim_{p\nearrow p_0} \mathbb{P}^*[0 \in C_p]$$

note $\{0 \in C_p\} \subset \{0 \in C_{p'}\}$ for all $p < p'$

so above is $\mathbb{P}^*\left[\bigcup_{p<p_0} \{0 \in C_p\}\right]$

$= \mathbb{P}^*[0 \in C_p \text{ for some } p < p_0]$

now $\theta(p_0) - \lim_{p \nearrow p_0} \theta(p) = \mathbb{P}^*[0 \in C_{p_0}] - \mathbb{P}^*[0 \in C_p \text{ for some } p < p_0]$

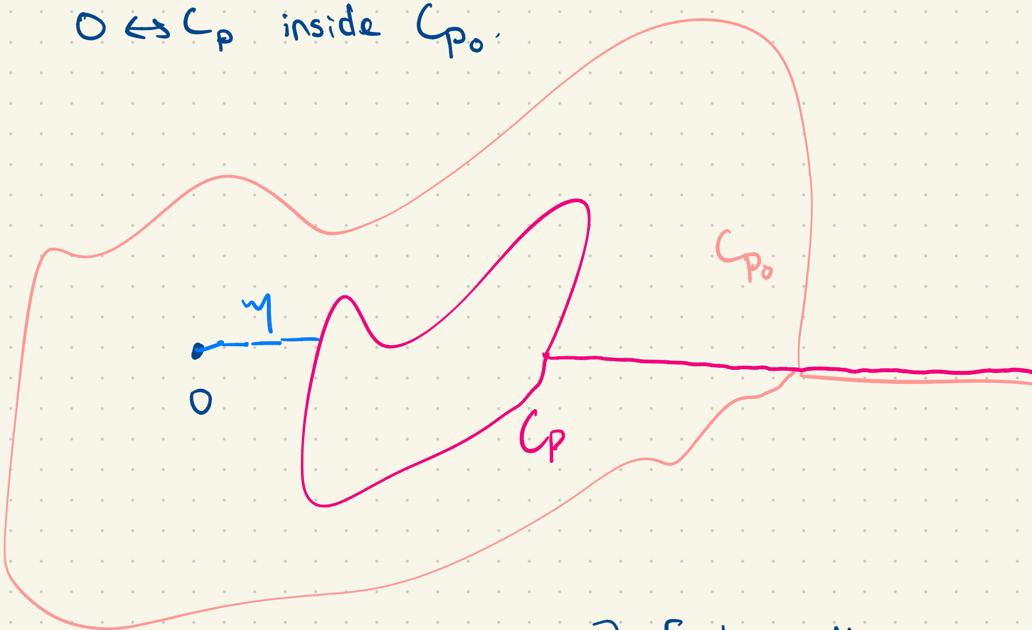
note $\{0 \in C_{p_0}\} \supset \{0 \in C_p \text{ for some } p < p_0\}$, so

above $= \mathbb{P}^*[\{0 \in C_{p_0}\} \setminus \{0 \in C_p \text{ for some } p < p_0\}]$

$= \mathbb{P}^*[0 \in C_{p_0},\ 0 \notin C_p\ \forall p < p_0]$

let $p < p_0$. since ∞ cluster is unique,

$0 \leftrightarrow C_p$ inside C_{p_0}.



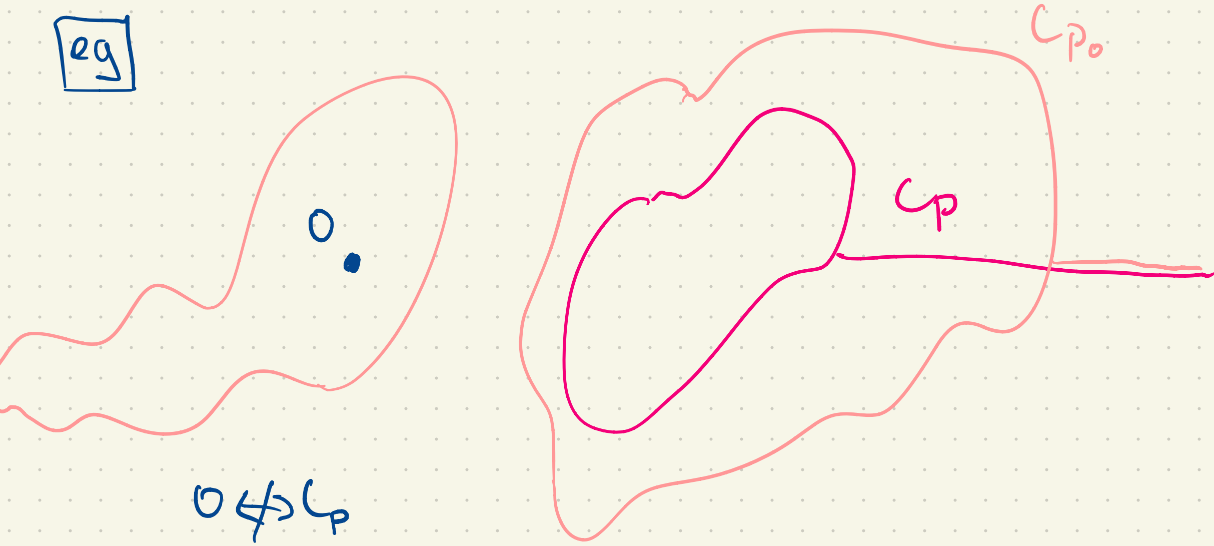
so $\exists$ finite path γ $0 \leftrightarrow C_p$ st. $\forall e \in \gamma$, $U_e \geq 1-p_0$ and at least one $e \in \gamma$ has $U_e = 1-p_0$

so $\theta(p_0) - \lim_{p \nearrow p_0} \theta(p) \leq \mathbb{P}^*[\exists e \in E(\mathbb{Z}^d) : U_e = 1-p_0]$

$$\underset{\text{union bound}}{\leq} \sum_{e\in E(\mathbb{Z}^d)} \mathbb{P}^*\left[U_e = 1-p_0\right]$$

$$= \sum_{e\in E(\mathbb{Z}^d)} 0 \underset{\substack{\text{as } E(\mathbb{Z}^d)\\ \text{is countable}}}{=} 0 \qquad \blacksquare$$

$\boxed{\text{eg}}$



$0 \not\leftrightarrow C_p$ inside C_{p_0} $\Rightarrow$ $\exists\ 2\ \infty$ clusters at p_0.