measure theory part 2 : independence & expectation.

$\boxed{*}$ in the lectures we do not give any proofs from this section — i include them here in case you're interested.

① independence

def let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, and let $A_i \in \mathbb{F}$ $\forall i \in I$, where I is any index set. we say $\{A_i\}_{i \in I}$ are <u>independent</u> if for all finite sets $J \subset I$,

$$\mathbb{P}\left[\bigcap_{i \in J} A_i\right] = \prod_{i \in J} \mathbb{P}[A_i].$$

def let $\mathbb{F}_i$ be a sub-σ-algebra of $\mathbb{F}$ $\forall i \in I$. (subsets of $\mathbb{F}$ which are σ-algebras). we say $\{\mathbb{F}_i\}_{i \in I}$ are independent if $\forall A_i \in \mathbb{F}_i$, $i \in I$, $\{A_i\}_{i \in I}$ are independent.

thm let $\mathbb{A}_1, \mathbb{A}_2$ satisfy $\emptyset \in \mathbb{A}_i$, $A, B \in \mathbb{A}_i \Rightarrow A \cap B \in \mathbb{A}_i$ $\forall i = 1, 2$. suppose

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]$$

$\forall A_1 \in \mathbb{A}_1$, $A_2 \in \mathbb{A}_2$. then $\sigma(\mathbb{A}_1), \sigma(\mathbb{A}_2)$ indep.

proof fix $A_1 \in \mathbb{A}_1$ and define forall $A \in \mathbb{F}$

$$\mu(A) := \mathbb{P}(A_1 \cap A), \quad \nu(A) := \mathbb{P}[A_1]\mathbb{P}[A]$$

then μ and ν are measures which agree on $\mathbb{A}_2$, with $\mu(\Omega) = \nu(\Omega) = \mathbb{P}[A_1] < \infty$. so by uniqueness of caratheodory, $\forall A_2 \in \sigma(\mathbb{A}_2)$,

$$\mathbb{P}[A_1 \cap A_2] = \mu(A_2) = \nu(A_2) = \mathbb{P}[A_1]\mathbb{P}[A_2].$$

—

now fix $A_2 \in \sigma(\mathbb{A}_2)$ and repeat the argument with

$$\mu'(A) := \mathbb{P}[A \cap A_2], \quad \nu'(A) := \mathbb{P}[A]\mathbb{P}[A_2]$$

$\blacksquare$

lem (exercises) let $(\Omega_1, \mathbb{F}_1)$, $(\Omega_2, \mathbb{F}_2)$ be sets with σ-algebras, and $X : \Omega_1 \to \Omega_2$ measurable. then

$$X^{-1}(\mathbb{F}_2) := \{ X^{-1}(A) : A \in \mathbb{F}_2 \} \text{ is a } \sigma\text{-algebra}.$$

we often write $\sigma(X) := X^{-1}(\mathbb{F}_2)$.

def $(X_i : i \in I)$ RVs on $(\Omega, \mathbb{F}, \mathbb{P})$ are **independent** if the σ-algebras $\sigma(X_i)$ are independent.

(2) expectation

- we will define, for measurable functions f on $(\Omega, \mathbb{F}, \mu)$ the integral of f

$$\mu(f) = \int_\Omega f \, d\mu .$$

when $(\Omega, \mathbb{F}) = (\mathbb{R}, \mathbb{B})$ and μ is Lebesgue measure, we write

$$\mu(f) = \int_{\mathbb{R}} f(x) \, dx .$$

for $(\Omega, \mathbb{F}, \mathbb{P})$ a probability space, $f = X$ a random variable, we call it the expectation of X

$$\mu(X) = \mathbb{E}[X].$$

def $f : \Omega \to \mathbb{R}$ is simple if

$$f = \sum_{k=1}^{m} a_k \mathbb{1}_{A_k} ,$$

where $0 \leq a_k < \infty$, $A_k \in \mathbb{F}$, $m \in \mathbb{N}$. for f simple, define

$$\mu(f) := \sum_{k=1}^{m} a_k \mu(A_k)$$

(setting $0 \cdot \infty = 0$).

def we say a property P holds almost everywhere (or almost surely in the probability setting) if $\mu(\{P \text{ doesn't hold}\}) = 0$.

lem for $\alpha, \beta \geq 0$, f, g simple,

a. $\mu(\alpha f + \beta g) = \alpha \mu(f) + \beta \mu(g)$

b. $f \leq g \implies \mu(f) \leq \mu(g)$

c. $f = 0$ a.e. $\iff \mu(f) = 0$

def if f is measurable & non-negative, let

$$\mu(f) := \sup\{\mu(g) : g \text{ simple}, \ g \leq f\}$$

- if f measurable, set $f^+ = f \vee 0$, $f^- = (-f) \vee 0$. then $f = f^+ - f^-$, $|f| = f^+ + f^-$ (all measurable).

def if $\mu(|f|) < \infty$, we say f is integrable and

$$\mu(f) := \mu(f^+) - \mu(f^-).$$

def let $x_n, n \in \mathbb{N}$ and x lie in $[0,\infty]$. we say $x_n \nearrow x$ if $x_n \leq x_{n+1}$ $\forall n$ and $x_n \to x$.

def let Ω be a set, $f_n, n \in \mathbb{N}$ and f non-negative functions on Ω. we say $f_n \nearrow f$ if $f_n(x) \nearrow f(x)$ $\forall x \in \Omega$.

def let $A_n, n \in \mathbb{N}$ and A be subsets of Ω. we say $A_n \nearrow A$ if $A_n \subset A_{n+1}$ $\forall n$ and $\bigcup_{n=1}^{\infty} A_n = A$.

thm (monotone convergence thm). let f, $(f_n)_{n \in \mathbb{N}}$ non-negative measurable, with $f_n \nearrow f$. then $\mu(f_n) \nearrow \mu(f)$.

proof

exercise.

<u>rmk</u> we have been using monotone convergence already.

$\boxed{\text{eg}}$ in chapter $\boxed{5}$, we $\mathbb{P}_p[\#\ \infty \text{ clusters} \geq 3] = 1$.

let $I_n = \{\geq 3\ \infty \text{ clusters intersecting } B_n\}$.

then $I_n \subset I_{n+1}$, and $\bigcup_{n=1}^{\infty} I_n = \{\exists \geq 3\ \infty \text{ clusters}\}$

hence $\lim_{n \to \infty} \mathbb{P}_p[I_n] = 1$ by monotone convergence

$\Rightarrow \exists n_0$ with $\mathbb{P}_p[I_{n_0}] > 0$.

<u>thm</u> for $\alpha, \beta \geq 0$, $f, g \geq 0$ and measurable

a. $\mu(\alpha f + \beta g) = \alpha \mu(f) + \beta \mu(g)$

b. $f \leq g \Rightarrow \mu(f) \leq \mu(g)$

c. $f = 0$ a.e. $\Leftrightarrow \mu(f) = 0$

<u>proof</u> • define simple functions

$$f_n = \min\{(2^{-n} \lfloor 2^n f \rfloor), n\} \quad \text{where}$$

$\lfloor x \rfloor$ is the largest integer k with $k \leq x$,

and g_n similar.

- then $f_n \nearrow f$, $g_n \nearrow f$, so $\alpha f_n + \beta g_n \nearrow \alpha f + \beta g$. hence by monotone convergence, $\mu(f_n) \nearrow \mu(f)$, $\mu(g_n) \nearrow \mu(g)$ and $\mu(\alpha f_n + \beta g_n) \nearrow \mu(\alpha f + \beta g)$.

- we obtain (a) from above and linearity for simple functions

- (b) from definition of the integral
- if $f = 0$ a.e. then $f_n = 0$ a.e., so $\forall n \in \mathbb{N}$, $\mu(f_n) = 0 \Rightarrow \mu(f) = 0$. on the other hand, if $\mu(f) = 0$ then $\mu(f_n) = 0$ $\forall n$, so $f_n = 0$ a.e. $\Rightarrow f = 0$ a.e.

$\blacksquare$

thm for $\alpha, \beta \in \mathbb{R}$, f, g integrable

a. $\mu(\alpha f + \beta g) = \alpha \mu(f) + \beta \mu(g)$

b. $f \leq g \Rightarrow \mu(f) \leq \mu(g)$

c. $f = 0$ a.e. $\boxed{\Rightarrow}$ $\mu(f) = 0$

proof • note $\mu(-f) = -\mu(f)$. for $\alpha \geq 0$,

$$\mu(\alpha f) = \mu(\alpha f^+) - \mu(\alpha f^-) = \alpha \mu(f^+) - \alpha \mu(f^-) = \alpha \mu(f)$$

- if $h = f+g$, then $h^+ + f^- + g^- = h^- + f^+ + g^+$

$$\Rightarrow \mu(h^+) + \mu(f^-) + \mu(g^-) = \mu(h^-) + \mu(f^+) + \mu(g^+)$$

$$\Rightarrow \mu(f) = \mu(f) + \mu(g)$$, so (a) holds.

- (b): if $f \leq g$ then $\mu(g) - \mu(f) = \mu(g-f) \geq 0$

- (c): $f = 0$ a.e. $\Rightarrow f^\pm = 0$ a.e.

$$\Rightarrow \mu(f^\pm) = 0 \quad \Rightarrow \mu(f) = 0.$$ ■

lem (Fatou's lemma) let $f_n, n \in \mathbb{N}$ be ≥ 0 and measurable.
then $\mu\left(\liminf_{n\to\infty} f_n\right) \leq \liminf_{n\to\infty} \mu(f_n)$.

(where recall $\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf_{m\geq n} \{x_m\}$

and $\left(\liminf_{n\to\infty} f_n\right)(x) := \liminf_{n\to\infty} \left(f_n(x)\right)$.)

proof
- forall $k \geq n$, $\inf_{m\geq n} f_m \leq f_k$

$$\Rightarrow \mu\left(\inf_{m\geq n} f_m\right) \leq \inf_{k\geq n} \mu(f_k) \leq \liminf_{n\to\infty} \mu(f_n).$$

$\boxed{*}$

- further, $\inf_{m \geq n} f_m \nearrow \liminf_{n \to \infty} f_n$

so by monotone convergence,

$$\mu\left(\inf_{m \geq n} f_m\right) \nearrow \mu\left(\liminf_{n \to \infty} f_n\right)$$

- hence by $\boxed{*}$ $\mu\left(\liminf_{n \to \infty} f_n\right) \leq \liminf_{n \to \infty} \mu(f_n)$.

$\blacksquare$

<u>thm</u> (dominated convergence theorem)

let $f_n, n \in \mathbb{N}$ and f be measurable. suppose $f_n \to f$ and $|f_n| \leq g$ $\forall n$, g some integrable function. then f and f_n are integrable $\forall n$, and

$$\mu(f_n) \to \mu(f).$$

<u>proof</u>
- f is measurable and $|f| \leq g$, so $\mu(|f|) \leq \mu(g) < \infty$, so f is integrable. similar for f_n.
- we have $0 \leq g \pm f_n \xrightarrow{n \to \infty} g \pm f$.

  so certainly $\liminf_{n \to \infty} (g \pm f_n) = g \pm f$

- by fatou,

$$\mu(g)+\mu(f) = \mu\left(\liminf_{n\to\infty}(g+f_n)\right) \leq \liminf_{n\to\infty} \mu(g+f_n) =$$
$$= \mu(g) + \liminf_{n\to\infty} \mu(f_n)$$

and

$$\mu(g)-\mu(f) = \mu\left(\liminf_{n\to\infty}(g-f_n)\right) \leq \liminf_{n\to\infty} \mu(g-f_n)$$
$$= \mu(g) - \limsup_{n\to\infty} \mu(f_n)$$

since $\mu(g)<\infty$, the above gives

$$\mu(f) \leq \liminf_{n\to\infty} \mu(f_n) \leq \limsup_{n\to\infty} \mu(f_n) \leq \mu(f)$$

so $\mu(f_n) \to \mu(f)$. ■

---

def let f, g be integrable on $(\Omega, \mathbb{F}, \mathbb{P})$.

- $\operatorname{cov}(f,g) := \mathbb{E}[fg] - \mathbb{E}[f]\,\mathbb{E}[g]$

- $\operatorname{var}(f) := \operatorname{cov}(f,f)$.