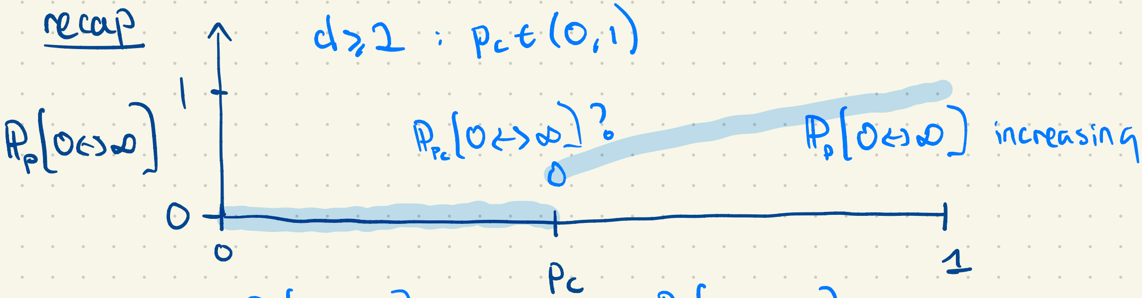
6. margulis-russo / covariance formula

recap $d \geq 2$: $p_c \in (0,1)$



$\mathbb{P}_p[0 \leftrightarrow \infty] = 0$ $\qquad$ $\mathbb{P}_p[0 \leftrightarrow \infty] > 0$

unique ∞ cluster

some remaining questions

- how does $\mathbb{P}_p[0 \leftrightarrow \infty]$ behave for $p > p_c$?
- for $p < p_c$, how big are the clusters ?

  In particular, how fast does $\mathbb{P}_p[0 \leftrightarrow \partial \Lambda_n] \searrow 0$ as $n \to \infty$?

rmk we are working towards a sharpness theorem :

- $\theta_n(p) \leq e^{-cn}$ $\quad \forall\, p < p_c$
- $\theta_n(p) \geq \frac{1}{1-p_c} \frac{p-p_c}{p}$ $\quad \forall\, p > p_c$.

this section gives a key tool.

motivation we want to study how $\theta(p)$, $\theta_n(p)$ behave as p varies. so it makes sense to differentiate in p !

prop (covariance formula) let $G = (V, E)$ be finite.
let $f : \{0,1\}^E \to \mathbb{R}$.
then

$$\frac{d}{dp} \mathbb{E}_p[f(\omega)] = \frac{1}{p(1-p)} \sum_{e \in E} \mathrm{cov}_p(f, \omega_e).$$

proof. $\mathbb{E}_p[f(\omega)] = \sum_{\omega \in \{0,1\}^E} f(\omega) p^{|\omega|} (1-p)^{|E|-|\omega|}$

note $\frac{d}{dp} p^{|\omega|} = \frac{|\omega|}{p} p^{|\omega|}$

$\frac{d}{dp} (1-p)^{|E|-|\omega|} = -\frac{|E|-|\omega|}{1-p} (1-p)^{|E|-|\omega|}$

so $\frac{d}{dp} \mathbb{E}_p[f(\omega)]$

$$= \sum_{\omega \in \{0,1\}^E} f(\omega) p^{|\omega|} (1-p)^{|E|-|\omega|} \underbrace{\left(\frac{|\omega|}{p} - \frac{|E|-|\omega|}{1-p} \right)}$$

$$= \frac{1}{p(1-p)} \left(|\omega|(1-p) - (|E|-|\omega|) p \right)$$

$$= \frac{1}{p(1-p)} \left(|\omega| - |E| p \right)$$

$$= \frac{1}{p(1-p)} \mathbb{E}_p\left[f(\omega) \cdot (|\omega| - |E| p) \right]$$

$$= \frac{1}{p(1-p)} \sum_{e\in E} \mathbb{E}_p\Big[f(\omega) \, (\omega_e - p) \Big]$$

$$= \frac{1}{p(1-p)} \sum_{e \in E} \Big(\mathbb{E}_p[f \cdot \omega_e] - \mathbb{E}[f]\,\mathbb{E}[\omega_e] \Big)$$

$$= \frac{1}{p(1-p)} \operatorname{cov}_p(f, \omega_e) \qquad \blacksquare$$

using $p = \mathbb{E}[\omega_e]$

- when $f = \mathbb{1}_A$ for A increasing, the covariance formula has a very nice intepretation.

(recall for $\omega, \omega' \in \{0,1\}^E$, $\omega \leq \omega'$ if $\omega_e \leq \omega'_e \ \forall e \in E$; A increasing if $\mathbb{1}_A$ increasing in this order.

ie: if A occurs & we open more edges, then we're still in A).

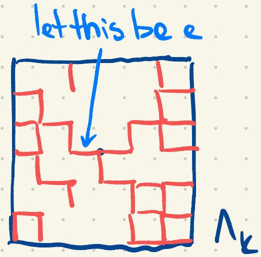
def let $A \subseteq \{0,1\}^E$ be increasing, and let $\omega \in \{0,1\}^E$. we say $e \in E$ is pivotal for A in ω if

$$\omega_{(e)} \notin A \quad \text{and} \quad \omega^{(e)} \in A$$

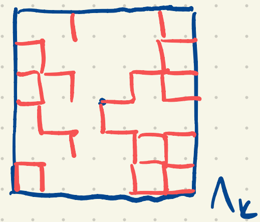
where $\omega_{(e)}(f) = \begin{cases} 0 & f = e \\ \omega(f) & \text{o/w} \end{cases}$, $\omega^{(e)}(f) = \begin{cases} 1 & f = e \\ \omega(f) & \text{o/w} \end{cases}$

note $\omega = \omega_{(e)}$ or $\omega^{(e)}$; if e is pivotal for A in ω, it's pivotal for A in the other one too.

eg let $A =$ 

, let $\omega =$ 

then $\omega = \omega^{(e)}$ and $\omega_{(e)}$ is

$\omega_{(e)} =$ 

then $\omega^{(e)} \in A$, $\omega_{(e)} \notin A$, so e is pivotal for A in $\omega = \omega^{(e)}$ and $\omega_{(e)}$.

def let $\{e \text{ pivotal for } A\} = \{\omega : e \text{ pivotal for } A \text{ in } \omega\}$
$\subseteq \{0,1\}^{E \setminus e}$.

note: in particular, ω_e and $\{e \text{ pivotal for } A\}$ independent.

prop (margulis 74, russo 78)
let $p \in [0,1]$, let $G = (V,E)$ finite, let $A \subseteq \{0,1\}^E$ increasing. then

$$\frac{d}{dp} \mathbb{P}_p[A] = \sum_{e \in E} \mathbb{P}_p[e \text{ pivotal for } A]$$

<u>proof</u> by the covariance formula,

$$\frac{d}{dp}\mathbb{P}_p[A] = \frac{1}{p(1-p)}\sum_{e\in E}\operatorname{cov}_p(\mathbb{1}_A, w_e).$$

$$= \frac{1}{p(1-p)}\sum_{e\in E}\mathbb{E}_p[\mathbb{1}_A\cdot(w_e-p)]$$

$$= \frac{1}{p(1-p)}\sum_{e\in E}\mathbb{E}_p\Big[\mathbb{1}_A(w_e-p)\big(\mathbb{1}_{e \text{ pivotal for } A} + \mathbb{1}_{e \text{ not pivotal for } A}\big)\Big]$$

if 2nd summand here gives 0, then we have just 1st summand. when e pivotal for A, $\mathbb{1}_A$ and $w_e=1$ are the same thing. so

$$= \frac{1}{p(1-p)}\sum_{e\in E}(1-p)\,\mathbb{E}_p\big[\mathbb{1}\{e \text{ pivotal for } A\}\mathbb{1}\{w_e=1\}\big]$$

$$= \frac{1}{p}\sum_{e\in E}p\cdot\mathbb{P}_p[e \text{ pivotal for } A]$$

as w_e and e pivotal for A are independent.

---

to get 2nd summand $=0$, notice that

$A\cap\{e \text{ not pivotal for } A\}\subset\{0,1\}^{E\setminus e}$, see diagram below

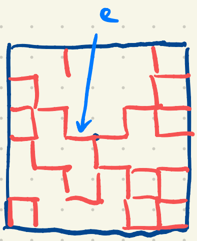
so independent from w_e, so summand is

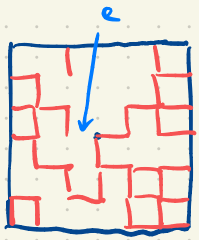
$$\frac{1}{p(1-p)} \sum_{e \in E} \mathbb{E}_p[w_e - p]\, \mathbb{E}_p\left[\mathbb{1}_A \mathbb{1}_{e \text{ not pivotal for } A} \right] = 0.$$

$$\left(\text{as } \mathbb{E}_p[w_e - p] = 0 \right).$$

∎

<u>eg</u>:

$w =$  Λ_k

$w =$  Λ_k

A occurs, and $\{e \text{ not pivotal for } A\}$ occurs regardless of w_e.