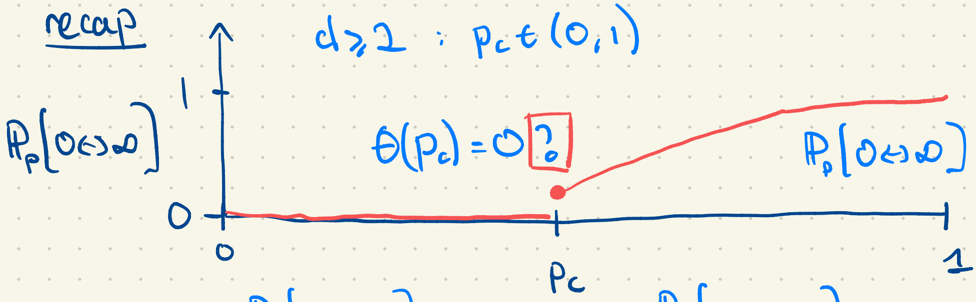
7. sharpness

recap



increasing, cts above p_c, right cts

$\mathbb{P}_p[0 \leftrightarrow \infty] = 0$

$\mathbb{P}_p[0 \leftrightarrow \infty] > 0$

unique ∞ cluster

thm (menshikov 86, aizenman, barsky 87)

1. $\forall\, p < p_c$, $\exists\, c = c(p) > 0$ st. $\forall k \geq 1$,

$$\theta_k(p) \leq e^{-ck}$$

"exponential decay"

2. $\forall\, p \geq p_c$,

$$\theta(p) \geq \frac{1}{1-p_c}\left(\frac{p-p_c}{p}\right)$$

rmk we expect for $p > p_c$ that $\theta(p) \sim (p-p_c)^{\beta(d)}$ for some $\beta = \beta(d) > 0$.

- expected that $\beta(d) = 1$ $\forall\, d \geq 6$
  this is proved for $d \geq 11$ (fitzner, hofstad 2017)
- for $2 \leq d \leq 5$, $\beta(d)$ is unique to d ;

only for $d=2$ is there a nice formula: $\beta(2) = \frac{5}{36}$.

proof (duminil-copin, tassion 16) (the fastest proof)

for $S \subset \mathbb{Z}^d$ a set of vertices, recall $0 \overset{S}{\leftrightarrow} x$ means 0 and x are connected using only vertices in S.

let $\Delta S = \{xy \in E : x \in S, y \notin S\}$. let

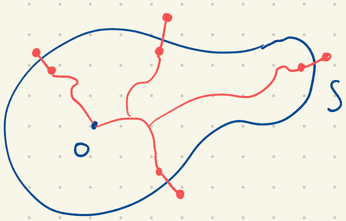
$$\phi_p(S) := p \cdot \sum_{xy \in \Delta S} \mathbb{P}_p[0 \overset{S}{\leftrightarrow} x]$$

set $\tilde{p}_c := \sup\{p \in [0,1] : \exists \text{ finite } S \ni 0 \text{ st. } \phi_p(S) < 1\}$

we will prove that the thm holds for $\tilde{p}_c$
(this automatically gives $p_c = \tilde{p}_c$).

note geometric interpretation of $\phi_p(S)$.

$$\phi_p(S) = \sum_{xy \in \Delta S} \mathbb{P}_p[\omega_{xy} = 1]\, \mathbb{P}_p[0 \overset{S}{\leftrightarrow} x]$$



$$= \sum_{xy \in \Delta S} \mathbb{P}_p[\omega_{xy} = 1, 0 \overset{S}{\leftrightarrow} x]$$

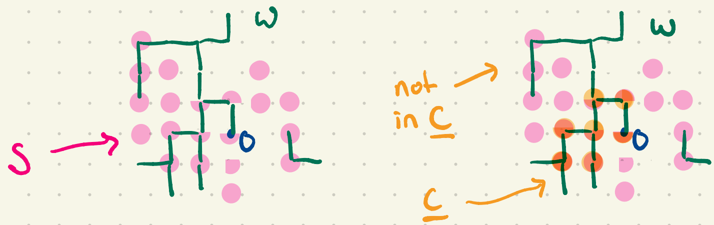
$$= \mathbb{E}_p\Big[\sum_{xy \in \Delta S} \mathbb{1}\{\omega_{xy} = 1, 0 \overset{S}{\leftrightarrow} x\}\Big]$$

$$= \mathbb{E}_p\left[\begin{array}{l}\text{number of edges through which} \\ \text{one can exit } S \text{ from } 0\end{array}\right]$$

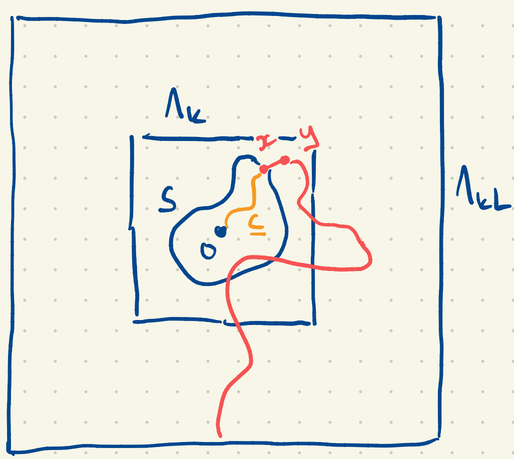
Part 1 if $p < \tilde{p}_c$ (ie $\exists$ such a finite S), then $\exists c_p > 0$ st. $\forall k \geq 1$, $\theta_k(p) \leq e^{-c_p k}$

proof let S finite, $0 \in S$, $\phi_p(S) < 1$. let k large enough st. $S \subset \Lambda_{k-1}$. let $L \in \mathbb{N}$, and assume $\{0 \leftrightarrow \partial \Lambda_{kL}\}$ holds. let

$$\underline{C} = \{ z \in S : 0 \overset{S}{\leftrightarrow} z \}$$ (random subset of S)



since $S \cap \partial \Lambda_{kL} = \phi$, $\exists$ edge $xy \in \Delta S$ st. $0 \overset{S}{\leftrightarrow} x$, $\omega_{xy} = 1$, $y \leftrightarrow \partial \Lambda_{kL}$ in $\underline{C}^c$



so $\{0 \leftrightarrow \partial \Lambda_{kL}\} \subset \bigcup_{xy \in \Delta S} \{0 \overset{S}{\leftrightarrow} x, \omega_{xy} = 1, y \overset{C^c}{\leftrightarrow} \partial \Lambda_{kL}\}$

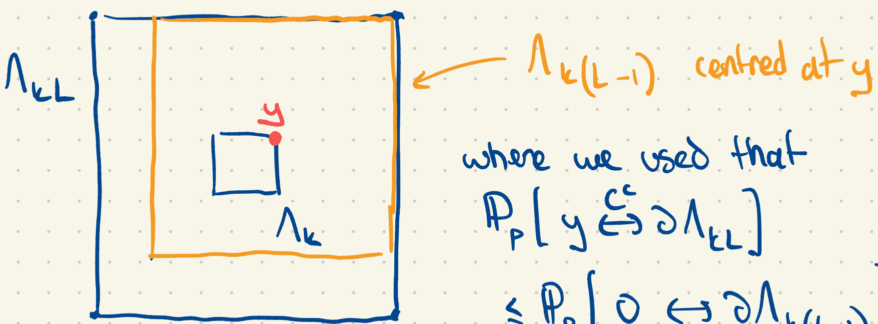
now $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{kL}]$

$$\leq \sum_{xy\in\Delta S} \sum_{C\subseteq S} \mathbb{P}_p\Big[\{0 \overset{S}{\leftrightarrow} x, \underline{C}=C\} \cap \{\omega_{xy}=1\} \cap \{y \overset{C^c}{\leftrightarrow} \partial\Lambda_{kL}\}\Big]$$

independence

$$= p \cdot \sum_{xy\in\Delta S} \sum_{C\subseteq S} \mathbb{P}_p[0 \overset{S}{\leftrightarrow} x, \underline{C}=C]\,\mathbb{P}_p[y \overset{C^c}{\leftrightarrow} \partial\Lambda_{kL}]$$

$$\leq p \cdot \sum_{xy\in\Delta S} \mathbb{P}_p[0 \overset{S}{\leftrightarrow} x] \cdot \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{k\cdot(L-1)}]$$



where we used that $\mathbb{P}_p[y \overset{C^c}{\leftrightarrow} \partial\Lambda_{kL}] \leq \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{k(L-1)}]$

so

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{kL}] \leq \phi_p(S) \cdot \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{k(L-1)}].$$

by induction, $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{kL}] \leq \phi_p(S)^{L-1} \cdot \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_k]$

$$\Rightarrow \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq c_1 \left[\phi_p(S)^{\frac{1}{k}}\right]^n$$

$$= e^{-cn} \quad \blacksquare$$

for some c_1, c.

part 2 $\quad \forall p > \tilde{p}_c, \quad \Theta(p) \geq \frac{1}{1-\tilde{p}_c} \cdot \frac{p-\tilde{p}_c}{p}$

2(a) : let $T_k = \{ x \in \Lambda_k : x \not\leftrightarrow \partial \Lambda_k \}$ (random subset of Λ_k). then

$$\frac{d}{dp} \Theta_k(p) = \frac{1}{p(1-p)} \mathbb{E}_p[\phi_p(T_k)]$$

2(b) $\quad \forall p > \tilde{p}_c \quad \frac{d}{dp} \Theta_k(p) \geq \frac{1}{p(1-p)} (1 - \Theta_k(p))$

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proof of 2(a)

let E_k be the edges of Λ_k. applying russo's formula to $A = \{ 0 \leftrightarrow \partial \Lambda_k \}$ gives

$$\frac{d}{dp} \Theta_k(p) = \sum_{e \in E_k} \mathbb{P}_p[e \text{ pivotal for } A]$$

$$= \frac{1}{1-p} \sum_{e \in E_k} \mathbb{P}_p[e \text{ pivotal for } A, \ \omega_e = 0]$$

ω_e, $\{e \text{ pivotal for } A\}$ indep

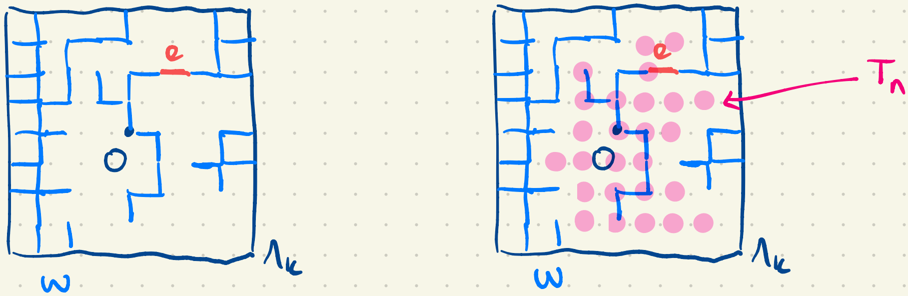
$$= \frac{1}{1-p} \sum_{e \in E_k} \mathbb{P}_p[e \text{ pivotal for } A, \ A^c]$$

note:

$$A^c = \{ 0 \in T_k \} = \bigcupdot_{\substack{T \subseteq \Lambda_k \\ 0 \in T}} \{ T_k = T \}$$

so above

$$= \frac{1}{1-p} \sum_{\substack{T \subseteq \Lambda_k \\ 0 \in T}} \sum_{e \in E_k} \mathbb{P}_p\left[e \text{ pivotal for } A,\ T_k = T \right]$$



recall $e = xy$ pivotal for $A = \{0 \leftrightarrow \partial\Lambda_k\}$ if $y \leftrightarrow \partial\Lambda_k$, $x \leftrightarrow 0$ and $0 \not\leftrightarrow \partial\Lambda_k$ in $E \setminus e$. adding that $A^c = \{0 \in T_k\} = \{0 \not\leftrightarrow \partial\Lambda_k\}$ occurs, with $T_k = T$ $e \in \Delta T$ and $x \overset{T}{\leftrightarrow} 0$. so above

$$= \frac{1}{1-p} \sum_{\substack{T \subseteq \Lambda_k \\ 0 \in T}} \sum_{\substack{e = xy \\ \in \Delta T}} \mathbb{P}_p\left[0 \overset{T}{\leftrightarrow} x,\ T_k = T \right]$$

notice $\{T_k = T\}$ dependent on edges with ≤ 1 vertex in T (indeed, $\{T_k = T\} = \{ \omega_e = 0 \ \forall e \in \Delta T \}$
$\cap \{ z \overset{T^c}{\leftrightarrow} \partial\Lambda_k \ \ \forall z \in T^c \}$).
whereas $\{0 \overset{T}{\leftrightarrow} x\}$ clearly dependent on edges with both vertices in T.

$$\overset{\text{indep}}{=} \frac{1}{1-p} \sum_{\substack{T \subseteq \Lambda_k \\ 0 \in T}} \sum_{\substack{e = xy \\ \in \Delta T}} \mathbb{P}_p\left[0 \overset{T}{\leftrightarrow} x \right] \cdot \mathbb{P}_p\left[T_k = T \right]$$

$$= \frac{1}{1-p} \sum_{\substack{T \subseteq \Lambda_k \\ 0 \in T}} \frac{1}{p} \varphi_p(T) \cdot \mathbb{P}_p\left[T_k = T \right]$$

$$= \frac{1}{p(1-p)} \mathbb{E}_p\left[\phi_p(T_k)\right] \qquad \blacksquare$$

proof of 2(b)

we want $\forall p > \tilde{p}_c \quad \frac{d}{dp}\theta_k(p) \geq \frac{1}{p(1-p)}(1-\theta_k(p))$.

we have $\frac{d}{dp}\theta_k(p) = \frac{1}{p(1-p)} \mathbb{E}_p\left[\phi_p(T_k)\right]$

since $p > \tilde{p}_c$, $\forall S$ finite with $0 \in S$, $\phi_p(S) \geq 1$, and if $0 \notin S$, then $\phi_p(S) = 0$. so $\phi_p(T_k) \geq \mathbb{1}\{0 \in T_k\}$

$$\geq \frac{1}{p(1-p)} \mathbb{E}_p\left[\mathbb{1}\{0 \in T_k\}\right]$$

$$= \frac{1}{p(1-p)} \mathbb{P}_p\left[0 \not\leftrightarrow \partial\Lambda_k\right] = \frac{1}{p(1-p)}(1-\theta_k(p)).$$

proof of part 2

rearranging $\quad \frac{d}{dp}\theta_k(p) \geq \frac{1}{p(1-p)}(1-\theta_k(p))$

gives $\quad \frac{d}{dp}\log\left(\frac{1}{1-\theta_k(p)}\right) \geq \frac{d}{dp}\log\left(\frac{p}{1-p}\right)$

(one can check this by hand). this holds $\forall p > \tilde{p}_c$.
integrating between $\tilde{p}_c$ and $p_1 > \tilde{p}_c$ gives

$$\log \frac{1-\theta_k(\tilde{p}_c)}{1-\theta_k(p_1)} \geq \log \frac{p_1}{1-p_1} \cdot \frac{1-\tilde{p}_c}{p}$$

$$\Rightarrow \quad \Theta_k(p_1) \geq 1 - \frac{1-p_1}{p_1}\,\frac{\tilde{p}_c}{1-\tilde{p}_c} = \frac{p_1 - p_1\tilde{p}_c - \tilde{p}_c + p_1\tilde{p}_c}{p_1(1-\tilde{p}_c)}$$

$$= \frac{1}{1-\tilde{p}_c}\,\frac{p_1-\tilde{p}_c}{p_1} \qquad \blacksquare$$