7.5 exponential decay in volume in subcritical regime

- last time we proved that $\forall p < p_c$, $\exists c > 0$ st.

$$\mathbb{P}_p[0 \leftrightarrow \partial \Lambda_k] \leq e^{-ck}$$

- let C_0 be the cluster of the origin (set of vertices). notice that if $|C_0| \geq (2k+1)^d = |\Lambda_k|$, then necessarily, $0 \leftrightarrow \partial\Lambda_k$. so

$$\mathbb{P}_p[|C_0| \geq (2k+1)^d] \leq e^{-ck}$$

$$\text{so} \quad \mathbb{P}_p[|C_0| \geq k] \leq e^{-c' k^{1/d}}$$

(stretched exponential decay)

- can we get a better bound?

thm (exponential decay in volume) let $p < p_c$. $\exists c > 0$ st. $\forall k \geq 1$,

$$\mathbb{P}_p[|C_0| \geq k] \leq e^{-ck}$$

lem let $A_n = \{C \subset \mathbb{Z}^d : 0 \in C,\ C \text{ connected},\ |C| = n\}$

"lattice animals of size n"

then $\forall n \geq 1$, $|A_n| \leq (4^{2d})^n$.

proof $\forall\, C \in A_n$, we have

$$\mathbb{P}_p[C_0 = C] \geq p^{|\{xy \in E \,:\, x,y \in C\}|} (1-p)^{|\Delta C|}$$

$$\geq p^{2dn} (1-p)^{2dn}$$

(at most 2d edges per vertex)

now, for $p = \frac{1}{2}$, we have

$$1 \geq \mathbb{P}_{\frac{1}{2}}\left[|C_0| = n\right] = \sum_{C \in A_n} \mathbb{P}_{\frac{1}{2}}\left[C_0 = C\right]$$

$$\geq |A_n| \left(\frac{1}{4}\right)^{2dn} \qquad \blacksquare$$

proof of thm

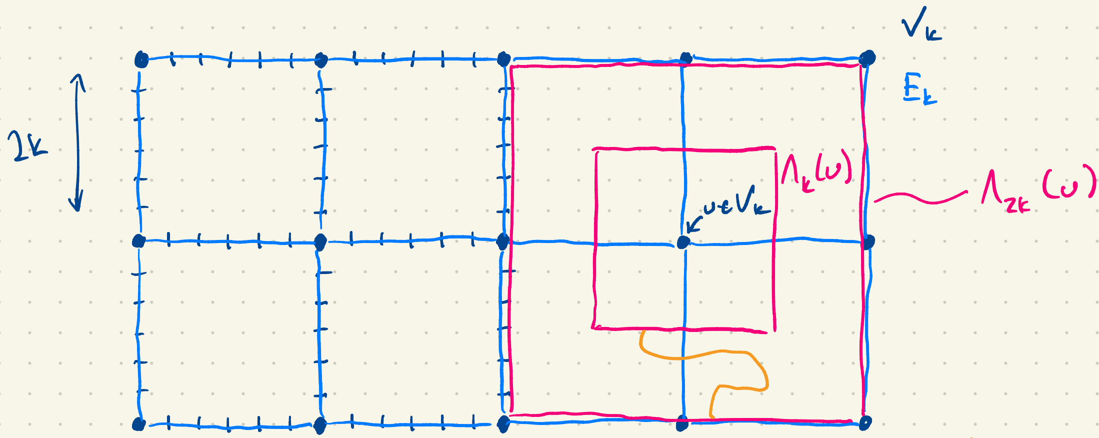
- we define a coarse-grained lattice $G_k = (V_k, E_k)$ and a site percolation on G_k.

  for $k \geq 1$, let $V_k := 2k\mathbb{Z}^d$

  $$E_k := \{\{2kx, 2ky\} : \|x-y\|_1 = 1\}$$

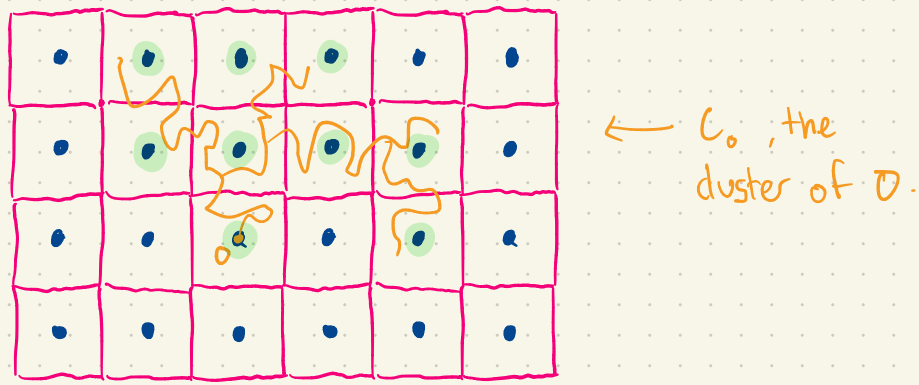
- consider a site percolation process $\eta \in \{0,1\}^{V_k}$ as:

  $$\eta(v) = \begin{cases} 1 & \text{if } \Lambda_k(v) \overset{\omega}{\longleftrightarrow} \partial\Lambda_{2k}(v) \\ 0 & \text{o/w} \end{cases}$$



if $\Lambda_k(v) \longleftrightarrow \partial\Lambda_{2k}(v)$

then $\eta(v) = 1$.



rmk

1. if $\|v-v'\|_\infty \geq 4k$, then $\eta(v)$, $\eta(v')$ indep. (and otherwise they are not indep!).

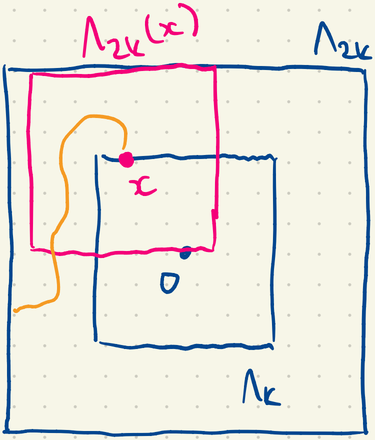
2. if $p<p_c$, using sharpness thm, we can make $\mathbb{P}_p[\eta(v)=1]$ arbitrarily small by increasing k.

so fix k large enough s.t.

$$\mathbb{P}_p[\eta(v)=1] = \mathbb{P}_p[\Lambda_k \leftrightarrow \partial\Lambda_{2k}] \leq \left(\frac{1}{4^{2d}e}\right)^{2^{d+1}} \qquad \boxed{*}$$

proof of $\boxed{*}$

$$\mathbb{P}_p[\Lambda_k \leftrightarrow \partial\Lambda_{2k}] = \mathbb{P}_p[\exists x \in \partial\Lambda_k : x \leftrightarrow \partial\Lambda_{2k}]$$



$$\leq \sum_{x \in \partial\Lambda_k} \mathbb{P}[x \leftrightarrow \partial\Lambda_{2k}]$$

$$\leq \sum_{x \in \partial\Lambda_k} \mathbb{P}[x \leftrightarrow \partial\Lambda_k(x)]$$

$$= |\partial\Lambda_k|\, \mathbb{P}[0 \leftrightarrow \partial\Lambda_k]$$

$$\leq C k^{d-1} e^{-ck}$$

$$= C' e^{-c'k}$$

for some $c', C' \in (0,\infty)$.