8. positive association / Harris-FKG inequality

- intuition: if we know that e is open, then this should help the event $x \leftrightarrow y$ occur. ie we expect

$$\mathbb{P}_p[x \leftrightarrow y \mid e \text{ open}] \geq \mathbb{P}_p[x \leftrightarrow y]$$

similarly, we expect

$$\mathbb{P}_p[x \leftrightarrow y \mid w \leftrightarrow z] \geq \mathbb{P}_p[x \leftrightarrow y].$$

"more edges only helps an increasing event"

thm (harris / fortuin-kasteleyn-ginibre inequality)

1. let A, B increasing events. then

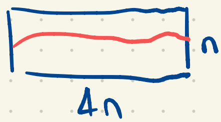
$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\,\mathbb{P}_p[B]$$

2. let $f, g : \Omega = \{0,1\}^E \to \mathbb{R}$ increasing, bounded & measurable (ie random variables) then

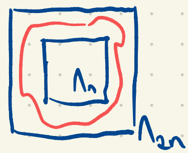
$$\mathbb{E}_p[f \cdot g] \geq \mathbb{E}_p[f] \cdot \mathbb{E}_p[g]$$

<u>rmk</u> this is a crucial property of bernoulli percolation, but also of many other percolation models.

<u>rmk</u> part 2 $\Rightarrow$ part 1, by setting $f = \mathbb{1}_A$, $g = \mathbb{1}_B$.

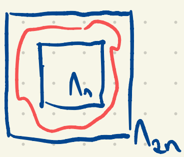
eg (in $\mathbb{Z}^2$) assume $\mathbb{P}_p\Big[$ 

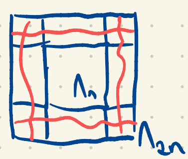
$\Big] \geq c > 0$.

then $\exists\, c' > 0$ st. $\mathbb{P}_p\Big[$ 

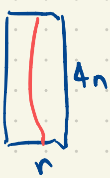
$\Big] \geq c'$.

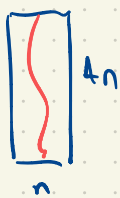
<u>proof</u>

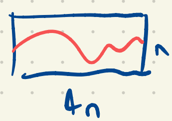
$\mathbb{P}_p\Big[$ 

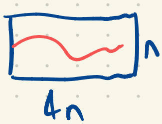
$\Big] \geq \mathbb{P}_p\Big[$ 

$\Big]$

$\geq \mathbb{P}_p\Big[$ 

$\Big] \cdot \mathbb{P}_p\Big[$ 

$\Big] \cdot \mathbb{P}_p\Big[$ 

$\Big] \cdot \mathbb{P}_p\Big[$ 

$\Big]$

$\geq c^4 > 0$. ∎

proof of thm

- define an ordering for $E(\mathbb{Z}^d)$, $E = \{e_1, e_2, \dots\}$ and write $\omega_i = \omega(e_i)$ $\forall i$.

1 finite volume

we prove the statement P_n:

$\forall f, g : \{0,1\}^n \to \mathbb{R}$ increasing,

$$\mathbb{E}_p[f(\omega_1, \dots, \omega_n)\, g(\omega_1, \dots, \omega_n)] \geq \mathbb{E}_p[f(\omega_1, \dots, \omega_n)]\, \mathbb{E}_p[g(\omega_1, \dots, \omega_n)]$$

$\boxed{n=1}$ • let $f, g : \{0,1\} \to \mathbb{R}$ increasing. adding a constant to f or g does not change the inequality, so WLOG, $f(0) = 0 = g(0)$.

- then $f(1) \geq 0$, $g(1) \geq 0$, e

$$\mathbb{E}_p[f(\omega_1)\,g(\omega_1)] - \mathbb{E}_p[f(\omega_1)]\,\mathbb{E}_p[g(\omega_1)]$$

$$= p\,f(1)\,g(1) - p^2 f(1)\,g(1) \geq 0.$$

$\boxed{\text{all } n}$ assume P_n holds. let $f, g : \{0,1\}^{n+1} \to \mathbb{R}$ increasing.

$$\mathbb{E}_p\left[f(\omega_1, \dots, \omega_{n+1})\, g(\omega_1, \dots, \omega_{n+1})\right]$$

$$= p\,\mathbb{E}_p\Big[\underbrace{f(\omega_1, \dots, \omega_n, 1)}\ \underbrace{g(\omega_1, \dots, \omega_n, 1)}\Big]$$

fn of n variables, increasing

$$+ (1-p)\,\mathbb{E}_p\Big[\overbrace{f(\omega_1, \dots, \omega_n, 0)}\ \overbrace{g(\omega_1, \dots, \omega_n, 0)}\Big]$$

using P_n

$$\geq p \cdot \underbrace{\mathbb{E}_p[f(\omega_1, \dots, \omega_n, 1)]}_{=: f_1(1)}\ \underbrace{\mathbb{E}_p[g(\omega_1, \dots, \omega_n, 1)]}_{=: g_1(1)}$$

$$+ (1-p)\,\underbrace{\mathbb{E}_p[f(\omega_1, \dots, \omega_n, 0)]}_{=: f_1(0)}\ \underbrace{\mathbb{E}_p[g(\omega_1, \dots, \omega_n, 0)]}_{=: g_1(0)}$$

$$= \mathbb{E}_p[f_1(\omega_1)\,g_1(\omega_1)] \underset{\text{using } P_1}{\geq} \mathbb{E}_p[f_1(\omega_1)]\,\mathbb{E}_p[g_1(\omega_1)]$$

this proves P_{n+1}, since

$$\mathbb{E}_p[f_1(\omega_1)] = p f_1(1) + (1-p) f_1(0)$$
$$= p\,\mathbb{E}_p[f(\omega_1,\dots,\omega_n,1)] + (1-p)\mathbb{E}_p[f(\omega_1,\dots,\omega_n,0)]$$
$$= \mathbb{E}[f(\omega_1,\dots,\omega_{n+1})]$$

& similar for g.

② infinite volume

- let $f, g : \{0,1\}^{E(\mathbb{Z}^d)} \to \mathbb{R}$ increasing, bounded random variables.

let $f_n = \mathbb{E}_p[f \mid \omega_1, \dots, \omega_n]$
$g_n = \mathbb{E}_p[g \mid \omega_1, \dots, \omega_n]$

we have not formally defined conditional expectation with respect to random variables. see end of chapter if you're interested.

- then $\forall n$, by P_n above,

$$\mathbb{E}_p[f_n g_n] \geq \mathbb{E}_p[f_n]\,\mathbb{E}_p[g_n]$$

- $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ are bounded martingales

by the martingale convergence theorem,

$f_n \to f$ and $g_n \to g$ $\mathbb{P}_p$-almost surely.

• Finally, using the dominated convergence theorem,

$$\mathbb{E}_p[f \cdot g] = \lim_{n\to\infty} \mathbb{E}_p[f_n \cdot g_n]$$

$$\geq \lim_{n\to\infty} \mathbb{E}_p[f_n]\,\mathbb{E}_p[g_n] = \mathbb{E}_p[f]\,\mathbb{E}_p[g]$$

$\blacksquare$

corollary 1. if A, B decreasing, then

$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\,\mathbb{P}_p[B]$$

2. if A increasing & B decreasing, then

$$\mathbb{P}_p[A \cap B] \leq \mathbb{P}_p[A]\,\mathbb{P}_p[B].$$

corollary (square root trick)

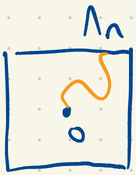
let $A_1, \dots, A_k$ increasing events, $\varepsilon > 0$.

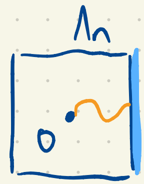
if $\mathbb{P}_p[A_1 \cup \dots \cup A_k] \geq 1 - \varepsilon$,

then $$\max_{1 \leq i \leq k} \{ \mathbb{P}_p[A_i] \} \geq 1 - \varepsilon^{1/k}$$

eg let $d=2$.

use rotation invariance

$\mathbb{P}_p\Big[$ 

$\Big] \geq 1-\varepsilon \implies \mathbb{P}_p\Big[$ 

$\Big] \geq 1-\varepsilon^{1/4}$

$\{0 \leftrightarrow \partial\Lambda_n\}$ $\qquad$ $\{0 \leftrightarrow$ right boundary of $\Lambda_n\}$

rmk "if the union of A_i is large, at least one of A_i is large"

proof we have

$$\mathbb{P}_p[A_1 \cup \cdots \cup A_k] = 1 - \mathbb{P}_p[A_1^c \cap \cdots \cap A_k^c]$$

$$\underset{\substack{\text{FKG} \\ \text{(several times)}}}{\leq} 1 - \mathbb{P}_p[A_1^c] \cdots \mathbb{P}_p[A_k^c]$$

$$\leq 1 - \Big(1 - \max_{1\leq i\leq k} \mathbb{P}_p[A_i]\Big)^k$$

$$\implies \max_{1\leq i\leq k} \mathbb{P}_p[A_i] \geq 1 - \Big(1 - \mathbb{P}_p[A_1 \cup \cdots \cup A_k]\Big)^{1/k}$$

$$\geq 1 - \varepsilon^{1/k}.$$

∎