9. BK – Reimer inequality

- the FKG inequality says that increasing events are positively correlated.
- the BK-Reimer inequality says: if we require events A and B to "occur on disjoint sets of edges" in $A \cap B$, then the events "become negatively correlated"

eg we will see that

$$\mathbb{P}_p\left[x \leftrightarrow y \text{ and } z \leftrightarrow w \text{ on two disjoint paths} \right]$$
$$\leq \mathbb{P}_p\left[x \leftrightarrow y \right] \mathbb{P}_p\left[z \leftrightarrow w \right]$$

- let's make this rigorous. let $G = (V, E)$ be any countable graph.

def let $A \subset \{0,1\}^E$ be an event and $\omega \in \{0,1\}^E$. a set $S \subset E$ is a witness of A for ω if $\omega \in A$ and $\forall \omega' \in \{0,1\}^E$ which agrees with ω on S, $\omega' \in A$ too.

def let $A, B \subset \{0,1\}^E$ be events.

$$A \circ B := \{ \omega \in \{0,1\}^E \quad : \quad \exists \text{ witnesses } I = I(\omega) \text{ of } A,$$

and $J = J(\omega)$ of B, $I \cap J = \emptyset \}$.

rmk
- we always have $A \circ B \subset A \cap B$
- if A, B are dependent on disjoint sets of edges, then $A \circ B = A \cap B$.
- lem: if A increasing & B decreasing, then

$$A \circ B = A \cap B.$$

thm (BK-Reimer inequality) let A, B be cylinder events. then

$$\mathbb{P}_p[A \circ B] \leq \mathbb{P}_p[A]\,\mathbb{P}_p[B]$$

rmk proved by van den Berg and Kesten (BK) 1985 for increasing events; extended by Reimer 2000 to all cylinder events.

proof (we only prove for increasing events).
- let $E = \{e_1, \dots, e_n\}$, $A, B \subset \{0,1\}^E$.
- we "duplicate" the edges by adding a second edge e_i' for each edge e_i.

ie:

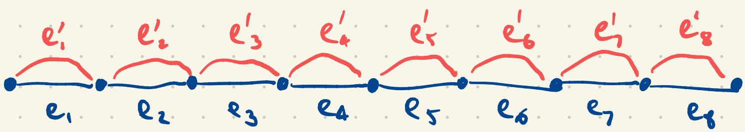
- let $E' = \{e'_1, \dots, e'_n\}$ and $\overline{E} = E \cup E'$. consider $\overline{\mathbb{P}}_p$, percolation measure on $\overline{E}$.

- we denote elements of $\{0,1\}^{\overline{E}}$ as $\overline{\omega}$. we then write

$$\omega = (\omega_1, \dots, \omega_n) \quad \text{where} \quad \omega_i = \overline{\omega}_{e_i}$$

and $$\omega' = (\omega'_1, \dots, \omega'_n) \quad \text{where} \quad \omega'_i = \overline{\omega}_{e'_i}.$$

$\boxed{\text{eg}}$

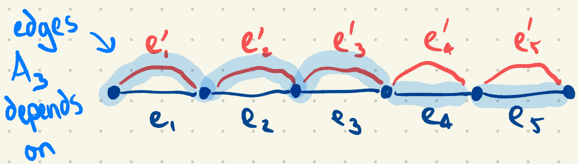


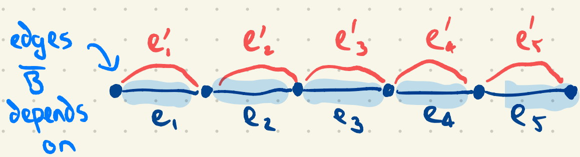
- for $0 \le i \le n$, let

$$\omega^{(i)} = (\omega'_1, \omega'_2, \dots, \omega'_i, \omega_{i+1}, \dots, \omega_n).$$

note $\omega^{(0)} = \omega$, $\omega^{(n)} = \omega'$.

- let $\overline{A}_i = \{\overline{\omega} : \omega^{(i)} \in A\}$, $\overline{B} = \{\overline{\omega} : \omega \in B\}$





then $\overline{\mathbb{P}}_p[\overline{A}_n \circ \overline{B}] = \overline{\mathbb{P}}_p[\overline{A}_n]\,\overline{\mathbb{P}}_p[\overline{B}] = \mathbb{P}_p[A]\,\mathbb{P}_p[B]$

(as $\overline{A}_n$, $\overline{B}$ depend on disjoint sets of edges).

and $$\overline{\mathbb{P}}_p\left[\overline{A}_0 \circ \overline{B}\right] = \mathbb{P}_p\left[A \circ B\right].$$

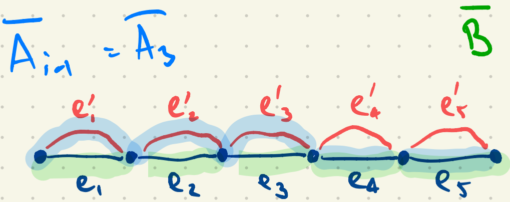
(as both $\overline{A}_0, \overline{B}$ are defined in terms of ω).

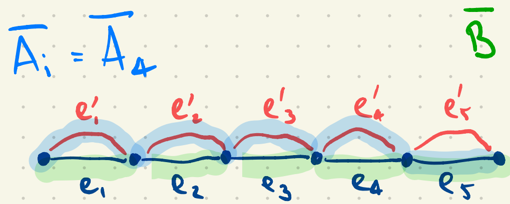
- hence BK-Reimer can be rewritten as

$$\overline{\mathbb{P}}_p\left[\overline{A}_0 \circ \overline{B}\right] \leq \overline{\mathbb{P}}_p\left[\overline{A}_n \circ \overline{B}\right] ,$$

meaning it suffices to prove

$$\boxed{*} \quad \overline{\mathbb{P}}_p\left[\overline{A}_{i-1} \circ B\right] \leq \overline{\mathbb{P}}_p\left[\overline{A}_i \circ B\right] \qquad \forall\, 1 \leq i \leq n.$$





rest is on ex sheet 5

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corollary on $\mathbb{Z}^d$, we have, for all $x, y, z, w \in \mathbb{Z}^d$,

$$\mathbb{P}_p\left[\{x \leftrightarrow y\} \circ \{z \leftrightarrow w\}\right] \leq \mathbb{P}_p\left[x \leftrightarrow y\right] \mathbb{P}_p\left[z \leftrightarrow w\right]$$

where we can set $w = \infty$, for example.

proof use BK-Reimer on $\{x \overset{\Lambda_n}{\longleftrightarrow} y\}$ and $\{z \overset{\Lambda_n}{\longleftrightarrow} w\}$

and take $n \to \infty$

for $w = \infty$, use $\{z \overset{\Lambda_n}{\longleftrightarrow} \partial\Lambda_n\}$ instead. $\blacksquare$