solutions for sheet 1.

$\boxed{1}$ (a) show $p_c = 1$ on $\mathbb{Z}$.

- clear that $\theta(1) = 1$.
- for $p \in [0,1)$,

$$\theta(p) = \mathbb{P}_p[0 \leftrightarrow \infty] = \mathbb{P}_p[\{0 \leftrightarrow +\infty\} \cup \{0 \leftrightarrow -\infty\}]$$

$$\underset{\text{bound}}{\overset{\text{union}}{\leq}} 2\mathbb{P}_p[0 \leftrightarrow +\infty]$$

now for all $L \in \mathbb{N}$, this is

$$\leq 2\mathbb{P}_p[\text{first } L \text{ edges are open}]$$

$$= 2p^L$$

so $\theta(p) \leq 2p^L \quad \forall L \in \mathbb{N}$

$$\Rightarrow \theta(p) = 0.$$

(b). similarly, $\theta(1)=1$, and

$$\theta(p) = \mathbb{P}_p[0 \leftrightarrow \infty] = \mathbb{P}_p\big[\{0 \leftrightarrow +\infty\} \cup \{0 \leftrightarrow -\infty\}\big]$$

$$\underset{\text{bound}}{\overset{\text{union}}{\leq}} 2\,\mathbb{P}_p[0 \leftrightarrow +\infty]$$

$\forall L \in \mathbb{N}$,

$$\mathbb{P}_p[0 \leftrightarrow +\infty] \leq \mathbb{P}_p\big[\forall x = 0, 1, \ldots, L-1, \ \exists i \in \{1, \ldots, n\} \text{ s.t. } \{(x,i),(x+1,i)\} \text{ open in } \omega\big]$$

$$= \mathbb{P}_p\Big[\bigcap_{x=0}^{L-1} \big\{\exists i \in \{1,\ldots,n\} \text{ s.t. } \{(x,i),(x+1,i)\} \text{ open in } \omega\big\}\Big]$$

independence

$$= \prod_{x=0}^{L-1} \mathbb{P}_p\Big[\big\{\exists i \in \{1,\ldots,n\} \text{ s.t. } \{(x,i),(x+1,i)\} \text{ open in } \omega\big\}\Big]$$

$$= \prod_{x=0}^{L-1} \left(1-(1-p)^n\right)$$

$$= \left(1-(1-p)^n\right)^L \to 0 \text{ as } L \to 0$$

so $\theta(p) = 0$.

(2) (a)

- $\Omega \in \mathcal{F}$ by defn.
- $\emptyset = \Omega^c \in \mathcal{F}$
- $B_i \in \mathcal{F} \quad \forall i \in \mathbb{N}$

$\Rightarrow B_i^c \in \mathcal{F} \quad \forall i \in \mathbb{N}$

$\Rightarrow \bigcup_{i=1}^{\infty} B_i^c \in \mathcal{F}$

$\Rightarrow \bigcap_{i=1}^{\infty} B_i = \left(\bigcup_{i=1}^{\infty} B_i^c\right)^c \in \mathcal{F}.$

(b)

- $\{x \leftrightarrow y\} = \bigcup_{n \geq 1} \bigcup_{\substack{\gamma \text{ path} \\ x \leftrightarrow y \\ \text{length } n}} \underbrace{\{\text{all edges in } \gamma \text{ open in } \omega\}}_{\text{cylinder sets}}$

or $= \bigcup_{n \geq 1} \overbrace{\{x \leftrightarrow y \text{ in } \Lambda_n\}}$

- $\{0 \leftrightarrow \infty\} = \bigcap_{n \geq 1} \{0 \leftrightarrow \partial \Lambda_n\}$

$\{x \leftrightarrow \infty\} = \bigcap_{n \geq 1} \{0 \leftrightarrow \partial \Lambda_n(x)\}$,

where $\partial \Lambda_n(x)$ is the box size n centred at x.

- $\{\exists \infty \text{ cluster}\} = \bigcup_{x \in \mathbb{Z}^d} \{x \leftrightarrow \infty\}$

we showed $\{x \leftrightarrow \infty\} \in \mathcal{F}$, so LHS $\in \mathcal{F}$ too.

(3) there was a mistake in this question.

(a) should read: let $\mathbb{D} = \{ A \in \mathbb{F} : \mu_1(A) = \mu_2(A) \}$.

show for $A, B \in \mathbb{D}$ such that $A \subseteq B$,

$B \setminus A \in \mathbb{D}$.

- $\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A)$

$$= \mu_2(B \setminus A).$$

(b)
$$\mu_1\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu_1\left(\bigcup_{i=1}^{\infty} (A_i \setminus A_{i-1})\right)$$

(setting $A_0 = \emptyset$)

$$= \sum_{i=1}^{\infty} \mu_1(A_i \setminus A_{i-1})$$

as $(A_i \setminus A_{i-1})$ all disjoint

$$= \sum_{i=1}^{\infty} \mu_2(A_i \setminus A_{i-1})$$

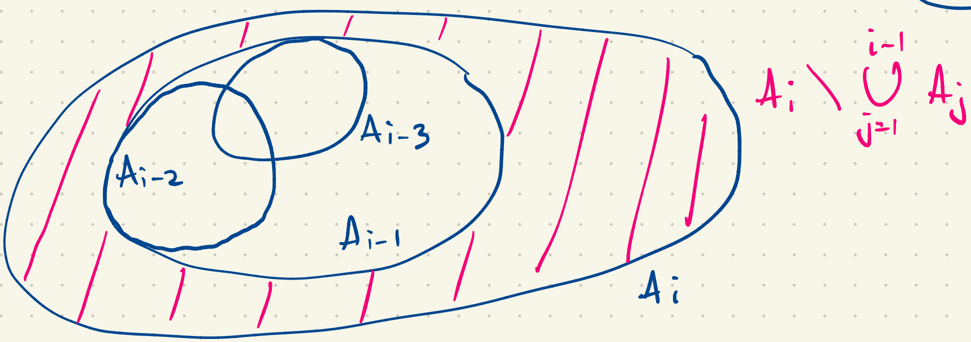
$$= \mu_2\left(\bigcup_{i=1}^{\infty} (A_i \setminus A_{i-1})\right)$$

$$= \mu_2\left(\bigcup_{i=1}^{\infty} A_i\right)$$

(c) should read: assuming additionally that $A, B \in \mathbb{D} \Rightarrow A \cap B \in \mathbb{D}$, show $\mathbb{D}$ is a σ-algebra.

suffices to show that for $A_i \in \mathbb{D}\ \forall i \in \mathbb{N}$, $\bigcup_{i=1}^{\infty} A_i \in \mathbb{D}$.

well, $\mu_1\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu_1\left(\bigcup_{i=1}^{\infty}\underbrace{\left(A_i \setminus \bigcup_{j=1}^{i-1} A_j\right)}_{\text{disjoint}}\right)$



$$= \sum_{i=1}^{\infty} \mu_1\left(A_i \setminus \bigcup_{j=1}^{i-1} A_j\right)$$

if $\mathbb{D}$ closed under complements and intersections, it's closed under finite unions $\Rightarrow$

$$= \sum_{i=1}^{\infty} \mu_2\left(A_i \setminus \bigcup_{j=1}^{i-1} A_j\right)$$

$$= \mu_2\left(\bigcup_{i=1}^{\infty} A_i\right) \quad \blacksquare$$

(4). $\mu(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(B_n) \; : \; B_n \in \mathcal{A} \;\; \forall n \in \mathbb{N}, \; A \subset \bigcup_{n=1}^{\infty} B_n \right\}$

- $\exists$ sequence $B_n \in \mathcal{A}$ st. $A \subset \bigcup_{n=1}^{\infty} B_n$

  and $\mu(A) \leq \sum_{n=1}^{\infty} \mu(B_n) + \frac{\varepsilon}{2}$

- for all $k \in \mathbb{N}$, using $A \subset \bigcup_{n=1}^{\infty} B_n$,

$$\mu\left(\left(\bigcup_{n=1}^{k} B_n\right) \setminus A\right) \leq \mu\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \setminus A\right)$$

$$= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) - \mu(A)$$

$$\overset{\text{union}}{\underset{\text{bound}}{\leq}} \sum_{n=1}^{\infty} \mu(B_n) - \mu(A) \leq \frac{\varepsilon}{2}$$

- since $A \subset \bigcup_{n=1}^{\infty} B_n$,

$$\mu\left(A \setminus \bigcup_{n=1}^{k} B_n\right) \to \mu(\emptyset) = 0$$

as $k \to \infty$

so for K large enough, $\mu\left(A \setminus \bigcup_{n=1}^{K} B_n\right) < \frac{\varepsilon}{2}$

- hence $\mu\left(A \,\Delta\, \bigcup_{n=1}^{K} B_n\right) < \varepsilon$.

$\blacksquare$

5 a let C be a lattice animal size n.

- let $\text{bulk}(C) = \{\text{edges } e = \{v_1, v_2\} \text{ s.t. } v_1, v_2 \in C\}$
  let $\Delta C = \{\text{edges } e = \{v_1, v_2\} \text{ s.t. } v_1 \in C, v_2 \notin C\}$.

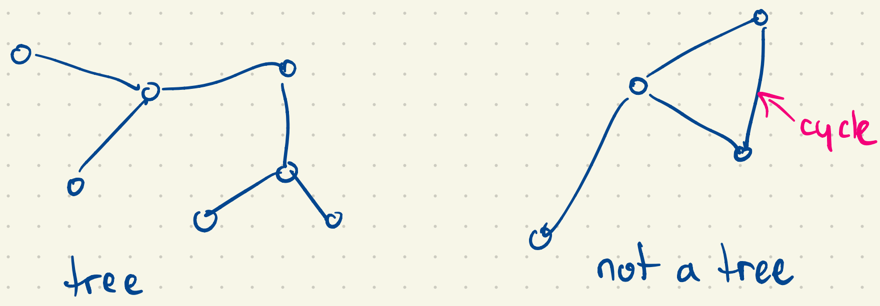
- $|\text{bulk}(C)| \leq 2dn$
  and $|\Delta C| \leq 2dn$

- $\mathbb{P}_p[C_0 = C] \geqslant \mathbb{P}_p[\text{all } e \in \text{bulk}(C) \text{ open and all } e \in \Delta C \text{ open}]$

$$\geqslant p^{2dn}(1-p)^{2dn}$$

- the above bound can be improved: (tuomas' solution)

let T_C be a spanning tree of C, that is, a set of edges in bulk (C), touching every vertex in C, such that there are no cycles in T_C. note $|T_C| = n-1$.



$$\mathbb{P}_p[C_0 = C] \geq \mathbb{P}_p[\text{edges of } T_C \text{ open, edges of } \Delta C \text{ closed}]$$

$$\geq p^{n-1}(1-p)^{2dn}.$$

(b). there was a typo : question should say show $|A_n| \leq 4^{2dn}$.

$$1 \geq \mathbb{P}_p[|C_0| = n] = \sum_{C \in A_n} \mathbb{P}[C_0 = C]$$

$$\geq \sum_{C \in |A_n|} p^{2dn}(1-p)^{2dn}$$

$$\geq |A_n| \left(\tfrac{1}{4}\right)^{2dn}.$$

- alternative proof by juho (?), giving better bound:

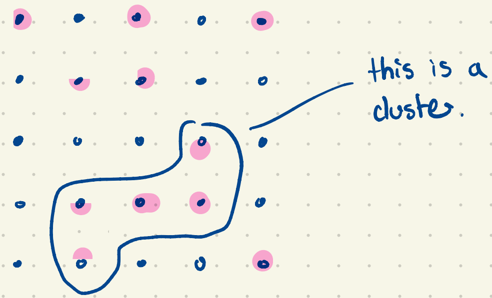
each T_C has a path of length $2(n-1)$ starting at 0, visiting every vertex of C, only using edges in T_C. so

$$|A_n| \leq \#\text{ paths starting at } 0, \text{ length } 2(n-1)$$

$$\leq (2d)^{2(n-1)}$$

⑥ (a). a definition of site percolation.

- a configuration is a map $\omega : \mathbb{Z}^d \to \{0,1\}$
- usual defn of open/closed vertices.
- a cluster in ω is a maximal connected component of open vertices in ω



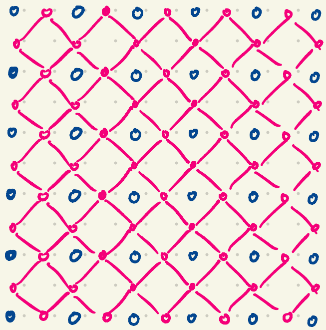
note not every vertex is in a cluster in this definition.

- we say $x \leftrightarrow y$ if x, y are open and they are in the same cluster.

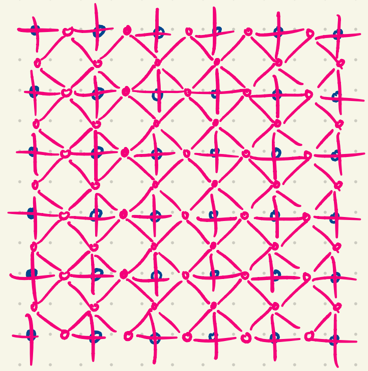
- the measure $\mathbb{P}_p^{\text{site}}$ is the product measure as defined for bond percolation.

(b)

$\mathbb{L}$



$\mathbb{L}^+$



- let $\mathbb{L}$ be the graph with vertex set $E(\mathbb{Z}^d)$,

  & edge set $\{\{e_1, e_2\} : e_1, e_2$ share 1 vertex and one face $\}$.

- let $\mathbb{L}^+$ be the graph with vertex set $E(\mathbb{Z}^d)$,

  & edge set $\{\{e_1, e_2\} : e_1, e_2$ share 1 vertex $\}$

- bond percolation on $\mathbb{Z}^2$ is equivalent to site percolation on $\mathbb{L}^+$ ($\exists$ measure-preserving bijection

  $$\{0,1\}^{E(\mathbb{Z}^2)} \leftrightarrow \{0,1\}^{V(\mathbb{L}^+)},$$

  preserving measure on all events in the σ-algebra)

- in particular, $\forall$ edges $e \in \mathbb{Z}^2$,

  $$\mathbb{P}_p^{\text{bond}}[e \leftrightarrow \infty] = \mathbb{P}_{p, \mathbb{L}^+}^{\text{site}}[e \leftrightarrow \infty]$$

(c) • $\mathbb{L}$ is a copy of $\mathbb{Z}^2$, and $\mathbb{L}$ is contained in $\mathbb{L}^+$. so $e \leftrightarrow \infty$ on $\mathbb{L} \Rightarrow e \leftrightarrow \infty$ on $\mathbb{L}^+$

• so $\mathbb{P}_p^{bond}[e \leftrightarrow \infty] \geq \mathbb{P}_p^{site}[e \leftrightarrow \infty]$ $\boxed{*}$

• in lectures we defined $\theta(p) = \mathbb{P}_p^{bond}[0 \leftrightarrow \infty]$.

let e be an edge of $\mathbb{Z}^2$ incident to 0. then

$$\mathbb{P}_p^{bond}[e \leftrightarrow \infty] = \mathbb{P}_p^{bond}[0 \leftrightarrow \infty, \ e \text{ open}]$$
$$\leq \mathbb{P}_p^{bond}[0 \leftrightarrow \infty]$$

combining with $\boxed{*}$ gives:

$$\Rightarrow p_c(\text{bond}) \leq p_c(\text{site}).$$