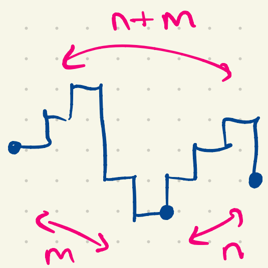
solutions for sheet 2

① (a) every SAW length $n+m$ is a union of a SAW length n and a SAW length m



this defines a map

$$C_{n+m} \to C_n \times C_m$$

which is injective but maybe not surjective.

so $c_{n+m} \leq c_n c_m$.

(b). $$\limsup_n \frac{\log c_n}{n} = \limsup_{m,r} \left\{ \frac{\log c_{km+r}}{km+r} \right\}$$

let $k \in \mathbb{N}$. then $n = km + r$

$$\leq \limsup_{m,r} \frac{m \log c_k + \log c_r}{km+r}$$

$$\leq \limsup_{m,r} \frac{\log c_k}{k} + \frac{\log c_r}{km}$$

$$= \frac{\log c_k}{k} + \limsup_{m,r} \frac{\log c_r}{km}$$

note that $1 \leq r \leq k$, so above is

$$\leq \frac{\log c_k}{k} + \limsup_m \frac{\log c_k}{km}$$

this limsup $\to 0$ as k is fixed.

so $\leq \frac{\log c_k}{k}$.

(c) since $\limsup_n \frac{\log c_n}{n} \leq \frac{\log c_k}{k}$ for all $k \in \mathbb{N}$,

$$\limsup_n \frac{\log c_n}{n} \leq \liminf_n \frac{\log c_n}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log c_n}{n} \text{ exists.}$$

taking exponentials gives $\lim_{n\to\infty} c_n^{1/n}$ exists $\in (0, \infty)$.

② (a) we showed $\theta(p)$ non-decreasing, so if $\theta(p) = 0$ then $\theta(p') = 0$ $\forall p' < p$, so $p \le p_c$.

(b) let $p > p'$. let $e_1, \dots, e_{2d}$ be the edges of $\mathbb{Z}^d$ incident to the origin. let v_1 be the vertex on e_1 which is not the origin.

$$\mathbb{P}^*\left[\, 0 \leftrightarrow \infty \text{ in } \omega \;,\; 0 \not\leftrightarrow \infty \text{ in } \omega' \,\right]$$

$$\geq \mathbb{P}^*\left[\, \underset{\text{outside of } e_1, \dots, e_{2d}}{v_1 \leftrightarrow \infty \text{ in } \omega} \;,\; \text{on } e_1, \dots, e_{2d} \;\; 1-p \le U_{e_i} < 1-p' \,\right]$$

this gives $0 \leftrightarrow v_1$ in ω, and $\omega'_{e_i} = 0$ $\forall i$, so $0 \not\leftrightarrow \infty$ in ω'

independence

$$= \mathbb{P}^*\left[\, \underset{\text{outside of } e_1, \dots, e_{2d}}{v_1 \leftrightarrow \infty \text{ in } \omega_p} \,\right] \cdot (p-p')^{2d}$$

$$\geq \mathbb{P}^*\left[\, v_1 \leftrightarrow \infty \text{ in } \omega_p \,\right] \cdot (p-p')^{2d}$$

$$= \theta(p) \cdot (p-p')^{2d} > 0 .$$

(c) using $\mathbb{P}^*[0 \not\leftrightarrow \infty \text{ in } \omega, \ 0 \leftrightarrow \infty \text{ in } \omega'] = 0,$

we have

$$\theta(p) = \mathbb{P}^*[0 \leftrightarrow \infty \text{ in } \omega]$$

$$= \mathbb{P}^*[0 \leftrightarrow \infty \text{ in } \omega \ \underline{\text{and}} \ \omega']$$

$$+$$

$$\mathbb{P}^*[0 \leftrightarrow \infty \text{ in } \omega, \ 0 \not\leftrightarrow \infty \text{ in } \omega']$$

$$\geq \mathbb{P}^*[0 \leftrightarrow \infty \text{ in } \omega']$$

$$+ \theta(p)(p-p')^{2d}$$

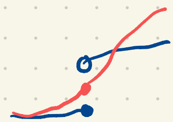
$$> \theta(p')$$

$\blacksquare$

3 Θ is right-continuous.

we know that $\Theta_k(p) = \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_k]$ is polynomial in p, so continuous, & is nondecreasing in p. moreover, $\Theta_k(p)$ is non-increasing in k. so $\Theta = \lim_{k\to\infty} \Theta_k$ is a "decreasing limit of $_{\text{cts}}$ increasing functions".

this gives right-continuity:



we want:

$$\lim_{x_n \downarrow x} \Theta(x_n) = \Theta(x)$$

let $y^+ = \text{LHS} = \lim_{x_n \downarrow x} \lim_{k\to\infty} \Theta_k(x_n)$,

$$y^- = \text{RHS} = \lim_{k\to\infty} \Theta_k(x).$$

we know that $\forall n \in \mathbb{N}$, $\forall k \in \mathbb{N}$, $\Theta_k(x_n) \geq y^+$ (*)

if $y^+ > y^-$ then $\exists k_0 \in \mathbb{N}$ st. $k > k_0$

$$\Rightarrow \Theta_k(x) < (y^+ + y^-)/2$$

now as Θ_k cts, $\exists n_0 \in \mathbb{N}$ st. $n \geq n_0$

$$\Rightarrow \Theta_k(x_n) < \frac{3y^+ + y^-}{4} < y^+$$

contradicts (*). ■

shorter proof by marlin

$$\lim_{x_n \searrow x} \Theta(x_n) - \Theta(x)$$

$$\leq \lim_{x_n \searrow x} \Theta_k(x_k) - \Theta(x)$$

$$= \Theta_k(x) - \Theta(x)$$ by continuity of Θ_k.

now $\lim_{x_n \searrow x} \Theta(x_n) - \Theta(x) \leq \Theta_k(x) - \Theta(x)$

for all k

$$\Rightarrow \lim_{x_n \searrow x} \Theta(x_n) - \Theta(x) \leq \lim_{k \to \infty} \Theta_k(x) - \Theta(x)$$

$$= 0$$

∎

④ (a) let $A, B \subset \Omega_2$.

- $f^{-1}(A \setminus B) = \{\omega \in \Omega_1 : f(\omega) \in A \setminus B\}$

$$= \{\omega \in \Omega_1 : f(\omega) \in A\} \setminus \{\omega \in \Omega_1 : f(\omega) \in B\}$$

$$= f^{-1}(A) \setminus f^{-1}(B)$$

- $f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \{\omega \in \Omega_1 : f(\omega) \in A_i \text{ for some } i\}$

$$= \bigcup_{i=1}^{\infty} \{\omega \in \Omega_1 : f(\omega) \in A_i\}$$

$$= \bigcup_{i=1}^{\infty} f^{-1}(A_i).$$

(b).(i) recall • $f^{-1}(\mathcal{A}) := \{f^{-1}(A) : A \in \mathcal{A}\}$

$$f^{-1}(\sigma(\mathcal{A})) := \{f^{-1}(A) : A \in \sigma(\mathcal{A})\}$$

- show $f^{-1}(\sigma(\mathcal{A}))$ is a σ-algebra:
  - $\Omega_2 \in \sigma(\mathcal{A})$, so $\Omega_1 = f^{-1}(\Omega_2) \in f^{-1}(\sigma(\mathcal{A}))$
  - let $f^{-1}(A), f^{-1}(B) \in f^{-1}(\sigma(\mathcal{A}))$.

    then $f^{-1}(A) \setminus f^{-1}(B) \underset{\text{part (a)}}{=} f^{-1}(A \setminus B) \in f^{-1}(\sigma(\mathcal{A}))$

  - let $f^{-1}(A_i) \in f^{-1}(\sigma(\mathcal{A}))$ $\forall i \in \mathbb{N}$.

then $\bigcup_{i=1}^{\infty} f^{-1}(A_i) = f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) \in f^{-1}(\sigma(\mathbb{A}))$

part (a)

so $f^{-1}(\sigma(\mathbb{A}))$ is a σ-algebra.

- $\mathbb{A} \subset \sigma(\mathbb{A}) \Rightarrow f^{-1}(\mathbb{A}) \subset f^{-1}(\sigma(\mathbb{A}))$.

  so $f^{-1}(\sigma(\mathbb{A}))$ is a sigma-algebra containing $f^{-1}(\mathbb{A})$

  $\Rightarrow \sigma(f^{-1}(\mathbb{A})) \subset f^{-1}(\sigma(\mathbb{A}))$.

(b)(ii)

let $\mathbb{D} = \{A \subset \Omega_2 : f^{-1}(A) \in \sigma(f^{-1}(\mathbb{A}))\}$

- let $A \in \mathbb{A}$. then $f^{-1}(A) \in f^{-1}(\mathbb{A}) \subset \sigma(f^{-1}(\mathbb{A}))$

  so $\mathbb{A}_2 \subset \mathbb{D}$.

- for $A \in \mathbb{D}$, $f^{-1}(\Omega_2 \setminus A) = \Omega_1 \setminus f^{-1}(A) \in \sigma(f^{-1}(\mathbb{A}))$.

  part (a)

- lets assume $A_i \in \mathbb{D}$, so $f^{-1}(A_i) \in \sigma(f^{-1}(\mathbb{A}))$

  then $f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i) \in \sigma(f^{-1}(\mathbb{A}))$

  part(a)

so $\mathbb{D}$ is a σ-algebra containing $\mathbb{A}$

$\Rightarrow \sigma(\mathbb{A}) \subset \mathbb{D}$.

- hence $f^{-1}(\sigma(\mathbb{A})) \subset \sigma(f^{-1}(\mathbb{A}))$ by defn of $\mathbb{D}$

(c) we want:

$$\forall A \in \sigma(\mathbb{A}_2), \quad f^{-1}(A) \in \sigma(\mathbb{A}_1).$$

or in other words,

$$f^{-1}(\sigma(\mathbb{A}_2)) \subset \sigma(\mathbb{A}_1).$$

well, $f^{-1}(\sigma(\mathbb{A}_2)) \underset{\text{part (b)}}{=} \sigma(f^{-1}(\mathbb{A}_2))$

now by assumption, $f^{-1}(\mathbb{A}_2) \subset \mathbb{A}_1$

so $\sigma(f^{-1}(\mathbb{A}_2)) \subset \sigma(\mathbb{A}_1)$

which gives the result. $\blacksquare$