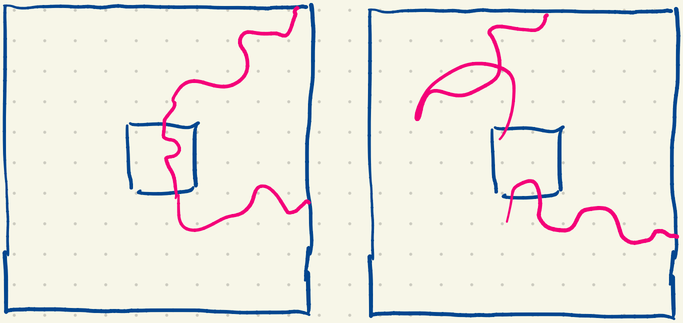
exercises 3 solutions

1. $K = \#$ disjoint clusters in Λ_n intersecting both Λ_k, $\partial\Lambda_n$

eg



$K=1$ $\qquad$ $K=2$ $\qquad$ $K=1$

$U_{k,n} := \{K \leq 1\}$.

(a) $U_{k,n} \subset U_{k,n+1}$. indeed,

if ω has K' disjoint clusters in Λ_{n+1} joining Λ_k and $\partial\Lambda_{n+1}$, then each of those clusters joins Λ_k to $\partial\Lambda_n$ in Λ_n.

such a cluster $\Lambda_k \leftrightarrow \partial\Lambda_{n+1}$ <u>may</u> split into more than one cluster $\Lambda_k \leftrightarrow \partial\Lambda_n$

hence $K(\omega) \geq K'(\omega)$

hence $K(\omega) \leq 1 \Rightarrow K'(\omega) \leq 1$.

$$\bigcup_{n=1}^{\infty} U_{k,n} = \{\#\ \infty \text{ clusters touching } \partial\Lambda_k \text{ is } \leq 1\}$$

$$\supseteq \{\#\ \infty \text{ clusters is } \leq 1\}$$

so $\mathbb{P}_p\left[\bigcup_{n=1}^{\infty} U_{k,n}\right] = 1.$

now $\lim_{n\to\infty} \mathbb{P}_p\left[U_{k,n}\right] = \mathbb{P}_p\left[\bigcup_{n=1}^{\infty} U_{k,n}\right] = 1.$

(monotone convergence).

(b). fix $\varepsilon > 0$, $k \in \mathbb{N}$.

$$O_m = \{p \in [0,1] : \mathbb{P}_p[U_{k,m}] > 1 - \varepsilon\}$$

- O_m is open, and we know $\mathbb{P}_p[U_{k,m}] \to 1$ as $m \to \infty$, so

$$[0,1] = \bigcup_{m \geq k} O_m \quad \text{an open cover of } [0,1]$$

by compactness, $\exists$ finite subcover:

$$[0,1] = \bigcup_{m \in \{m_1, \ldots, m_r\}} O_m .$$

let $n = \max_{1 \le i \le r} \{m_i\}$. then as $O_m \subseteq O_{m+1}$ (since $\mathbb{P}_p[U_{k,m}]$ nondecreasing in m), we have

$$[0,1] = O_n ,$$

which is what we wanted.

(c) let $p_1 > p_c$. we show $\theta_n \to \theta$ uniformly on $[p_1, 1]$.

- $\mathbb{P}_{p_1}[\Lambda_k \leftrightarrow \infty] \to 1$ as $k \to \infty$

  so let k large enough st. $\mathbb{P}_{p_1}[\Lambda_k \leftrightarrow \infty] \geq 1-\varepsilon$

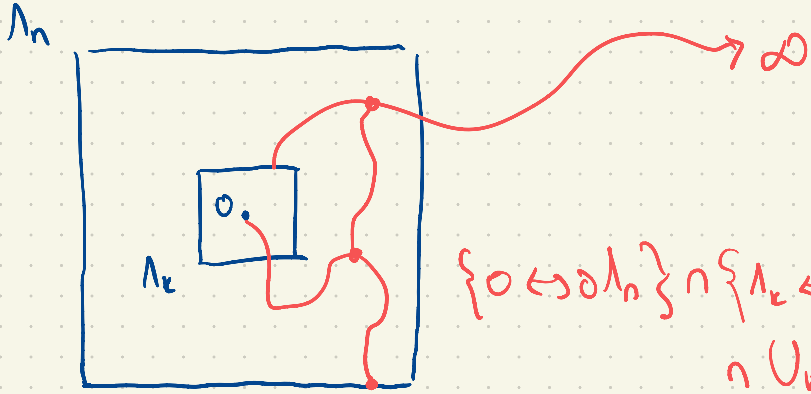
- by part (b), let $n \geq k$ st. $\forall p \in [0,1]$,

  $\mathbb{P}_p[U_{k,n}] \geq 1-\varepsilon$.

- then $\forall p \geq p_1$,

$$\theta_n(p) \geq \theta(p) \geq \mathbb{P}\left[\{0 \leftrightarrow \partial\Lambda_n\} \cap \{\Lambda_k \leftrightarrow \infty\} \cap U_{k,n}\right]$$

$$\geq \theta_n(p) - 2\varepsilon$$



$\{0 \leftrightarrow \partial\Lambda_n\} \cap \{\Lambda_k \leftrightarrow \infty\}$
$\cap\, U_{k,n}$

so $|\theta_n - \theta| \le 2\varepsilon$ on $[p_1, 1]$, which gives uniform convergence

$\blacksquare$

2.

thm (monotone convergence thm). let f, $(f_n)_{n\in\mathbb{N}}$ non-negative measurable, with $f_n \nearrow f$. then $\mu(f_n) \nearrow \mu(f)$.

proof case $\boxed{1}$ $f_n = \mathbb{1}_{A_n}$, $f = \mathbb{1}_A$.

use countable additivity; we must have $A_n \subset A_{n+1}$ $\forall n\in\mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n = A$. so $A = \bigcup_{n=1}^{\infty} A_n \setminus A_{n-1}$ (setting $A_0 = \emptyset$), and $A_n \setminus A_{n-1}$ all disjoint.

now $\mu(f) = \mu(A) = \sum_{n=1}^{\infty} \mu(A_n \setminus A_{n-1}) = \sum_{n=1}^{\infty} \mu(A_n) = \lim_{n\to\infty} \mu(f_n)$.

case $\boxed{2}$ f_n simple, $f = \mathbb{1}_A$.

fix $\varepsilon > 0$ and let $A_n = \{x : f_n(x) > 1-\varepsilon\}$

then $A_n \nearrow A$, and $(1-\varepsilon) \leq f_n \leq \mathbb{1}_A$.

$\Rightarrow$ $(1-\varepsilon)\mu(A_n) \leq \mu(f_n) \leq \mu(A)$.

lemma above

but $\mu(A_n) \nearrow \mu(A)$ by case 1, and ε arbitrary, so $\mu(f_n) \nearrow \mu(f)$.

case $\boxed{3}$ f_n, f simple.

write $f = \sum_{k=1}^{m} a_k \mathbb{1}_{A_k}$, with $a_k > 0$

and A_k disjoint. then $f_n \nearrow f$ implies

$$a_k^{-1} \cdot \mathbb{1}_{A_k} \cdot f_n \nearrow \mathbb{1}_{A_k}$$

so by case 2,

$$\mu(f_n) \underset{\text{linearity}}{=} \sum_{k=1}^{m} \mu(\mathbb{1}_{A_k} f_n) \underset{\text{above}}{\nearrow} \sum_{k=1}^{m} a_k \mu(A_k) \underset{\text{linearity}}{=} \mu(f)$$

case $\boxed{4}$ <u>f_n simple, $f \geq 0$ measurable</u>

let g simple, $g \leq f$. then $f_n \nearrow f$ implies $\min\{f_n, g\} \nearrow g$. so by case 3,

$$\mu(f_n) \geq \mu(\min\{f_n, g\}) \nearrow \mu(g).$$

since g arbitrary, $\mu(f_n) \nearrow \mu(f)$.

case $\boxed{5}$ <u>$f_n, f \geq 0$ and measurable</u>

set $g_n = \min\{(2^{-n}\lfloor 2^n f_n \rfloor), n\}$, where

$\lfloor x \rfloor$ is the largest integer k with $k \leq x$.

g_n is simple and $g_n \leq f_n \leq f$, so

$$\mu(g_n) \leq \mu(f_n) \leq \mu(f).$$

now $f_n \nearrow f$ forces $g_n \nearrow f$, so by case 4,

$\mu(g_n) \nearrow \mu(f)$ which gives the result. $\blacksquare$

3. (a)

- $f_0(s) = \mathbb{E}[s^{Z_0}] = \mathbb{E}[s] = s$
- $f_1(s) = \mathbb{E}[s^{Z_1}] = \mathbb{E}[s^{L_{0,1}}] = \sum_{k=0}^{\infty} p_k s^k$
- $f_1'(s) = \sum_{k=1}^{\infty} k p_k s^{k-1}$ $\quad f_1'(0) = p_1 = \mathbb{P}(L_{0,1} = 1)$

  $\Rightarrow f_1'(1) = \sum_{k=1}^{\infty} k p_k = \mathbb{E}[L_{0,1}]$
- $f_1''(s) = \sum_{k=2}^{\infty} k(k-1) p_k s^{k-2} \geq 0$

in particular, f strictly convex if $p_0 + p_1 < 1$.

(b)

- statement holds for $n=1$.
- $f_{n+1}(s) = \mathbb{E}[s^{Z_{n+1}}] = \mathbb{E}\left[s^{\sum_{i=1}^{Z_n} L_{n,i}}\right]$

$$= \sum_{k=0}^{\infty} \mathbb{E}\left[s^{\sum_{i=1}^{k} L_{n,i}} \,\middle|\, Z_n = k\right] \mathbb{P}[Z_n = k]$$

$$= \sum_{k=0}^{\infty} \mathbb{E}\left[s^{\sum_{i=1}^{k} L_{n,i}}\right] \mathbb{P}[Z_n = k]$$

independence of $L_{n,i}$ and Z_n

$$= \sum_{k=0}^{\infty} \mathbb{E}\left[s^{L_{0,1}}\right]^k \mathbb{P}[Z_n = k]$$

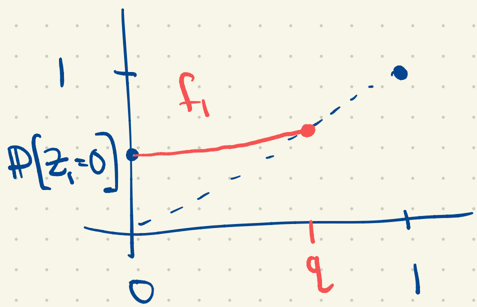
$\uparrow$ $L_{n,i}$ are iid

$$= \sum_{k=0}^{\infty} f_1(s)^k \, \mathbb{P}[Z_n = k]$$

$$= f_n(f_1(s)) .$$

(c) $q :=$ smallest fixed point of f_1 in $[0,1]$

- $\mathbb{P}[Z_n = 0] = f_n(0) = f_1 \circ \cdots \circ f_1(0)$.
- f_1 is a contraction on $[0,q]$, since $f_1(0) = \mathbb{P}[Z_1 = 0] \in [0,1]$, f_1 convex (if $f_1'(s) \geq 1$ for some $s < q$, then f_1 cannot then meet line $x = y$)



moreover since $f_1(1) = 1$, $q \in [0,1]$ exists.

- now by banach's fixed point theorem, as f_1 contraction on $[0,q]$ and q the ! fixed pt in

$[0,q]$, $\quad \underbrace{f_1 \circ \cdots \circ f_1}_{n \text{ times}}(0) \to q \quad$ as $n \to \infty$.

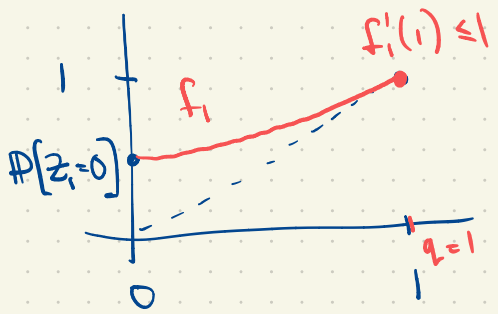
(d) $\mathbb{E}[L_{0,1}] = f_1'(1) > 1$

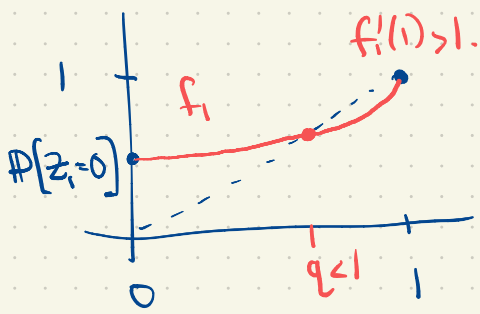
$\Rightarrow$ for ε small, $f_1(1-\varepsilon) < 1-\varepsilon$

$\Rightarrow$ by intermediate value thm, $\exists\, q \in [0, 1-\varepsilon]$ st.
$f(q) = q$.

- conversely, if $q<1$ (and excluding the degenerate case $p_1 = 1 \Rightarrow f_1(s) = s$ and $q=0$), we have that f_1 is strictly convex (as $p_1 + p_0 = 1 \Rightarrow q = 1$).

since $f_1(q) = q$, $f_1(1) = 1$, we have by mean value thm: $\exists\, r \in (q,1)$ st. $f_1'(r) = 1$. by strict convexity this means $\mathbb{E}[L_{0,1}] = f_1'(1) > 1$.





(e) consider $\mathbb{P}_p$ on our tree T_d. let C be the cluster of the root vertex.

let $Z_n = |C \cap \partial B_n(\text{root})|$ = # vertices in C at distance n from the root

- then Z_n is a branching process with offspring dist$\bar{n}$ (p_k) which is Binomial (d, p).

  hence $\mathbb{E}[L_{0,1}] = dp$.

- $|C| = \infty$ iff $Z_n > 0$ $\forall n$, ie if the branching process survives.

- so $\mathbb{P}_p[\text{root} \leftrightarrow \infty] > 0$

  $\Leftrightarrow \lim_{n\to\infty} \mathbb{P}_p[Z_n = 0] = q < 1$

  $\Leftrightarrow \mathbb{E}[L_{0,1}] = pd > 1$

  hence $p_c = \frac{1}{d}$. ∎