① ⓐ let $\delta_i = (0, \dots, 0, \delta, 0, \dots, 0)$ where $\delta > 0$.

$\uparrow$ i^{th} position

then $\frac{\partial}{\partial p_i} \mathbb{P}^*[\omega^{\underline{p}} \in A]$

$$= \lim_{\delta \to 0} \frac{\mathbb{P}^*[\omega^{\underline{p}+\delta_i} \in A] - \mathbb{P}^*[\omega^{\underline{p}} \in A]}{\delta}$$

now since $\omega^{\underline{p}} \leq \omega^{\underline{p}+\delta_i}$ (similar to $\omega^{(p)} \leq \omega^{(p')}$ in lectures),
and A increasing,
we have $\{\omega^{\underline{p}} \in A\} \subset \{\omega^{\underline{p}+\delta_i} \in A\}$.

$$\Rightarrow \mathbb{P}^*[\omega^{\underline{p}+\delta_i} \in A] - \mathbb{P}^*[\omega^{\underline{p}} \in A]$$

$$= \mathbb{P}^*[\omega^{\underline{p}+\delta_i} \in A, \omega^{\underline{p}} \notin A]$$

$$= \mathbb{P}^*[e_i \text{ pivotal for } A \text{ in } \omega^{\underline{p}}, \ 1-(p+\delta) \leq U_{e_i} < 1-p]$$

now recall that $\{e_i \text{ pivotal for } A \text{ in } \omega^{\underline{p}+\delta_i}\}$ and U_{e_i} are independent, so above is

$$= \mathbb{P}^*[e_i \text{ pivotal for } A \text{ in } \omega^{\underline{p}}] \cdot \mathbb{P}^*[1-(p+\delta) \leq U_{e_i} < 1-p]$$

$$= \mathbb{P}_p[e_i \text{ pivotal for } A] \cdot \delta.$$

$$\Rightarrow \lim_{\delta \to 0} \frac{\mathbb{P}^*[\omega^{\underline{p}+\delta_i} \in A] - \mathbb{P}^*[\omega^{\underline{p}} \in A]}{\delta}$$

$$= \mathbb{P}_p[e_i \text{ pivotal for } A]$$

(b) $$\frac{d}{dp} \mathbb{P}_p[A] = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \mathbb{P}^*[\omega^{\underline{p}} \in A]$$

$$= \sum_{i=1}^{n} \mathbb{P}_p[e_i \text{ pivotal for } A].$$

② (a) $\frac{1}{1-p} \mathbb{P}_p[A, w(e)=0]$

$$= \frac{1}{1-p} \sum_{\substack{w \in A \\ w(e)=0}} p^{|w|}(1-p)^{|E|-|w|}$$

where $|w| = |\{e \in E : w_e = 1\}|$

for all $w \in A$ with $w(e)=0$,
we know $\eta = w^e \in A$ (w^e agrees with w on $E \setminus \{e\}$ and $w^e(e)=1$) as A increasing.

= this map $w \to \eta$
maps $\{w \in A$ with $w(e)=0\}$
to $\{w \in A$ with $w(e)=1\}$
injectively, and gives above

$$= \frac{1}{1-p} \sum_{\substack{\eta \in A \\ \eta(e)=1 \\ \eta_e(e)=0}} p^{|\eta|}(1-p)^{|E|-|\eta|} \cdot \frac{1-p}{p}$$

since we changed the value of w on e.

$$\leq \frac{1}{p} \sum_{\substack{\eta \in A \\ \eta(e)=1}} p^{|\eta|}(1-p)^{|E|-|\eta|}$$

$$= \frac{1}{p} \mathbb{P}_p[w \in A, w(e)=1]$$

now above gives

$$\mathbb{P}_p[A \mid \omega(e)=1] \geq \mathbb{P}_p[A \mid \omega(e)=0]$$

now $\mathbb{P}_p[A] = p\,\mathbb{P}_p[A \mid \omega(e)=1]$

$+ (1-p)\mathbb{P}_p[A \mid \omega(e)=0]$

$\leq p\,\mathbb{P}_p[A \mid \omega(e)=1]$

by above

$+ (1-p)\mathbb{P}_p[A \mid \omega(e)=1]$

$= \mathbb{P}_p[A \mid \omega(e)=1]$

(b) $$\mathbb{P}_p[w_e=1 \mid A, C] = \frac{\mathbb{P}_p[w_e=1, A, C]}{\mathbb{P}_p[A, C]}$$

$$= \frac{\mathbb{P}_p[A \mid w_e=1, C]\,\mathbb{P}_p[w_e=1, C]}{\mathbb{P}_p[A \mid C]\,\mathbb{P}_p[C]}$$

$$= \frac{\mathbb{P}_p[A \mid w_e = 1, C]\, \mathbb{P}_p[w_e = 1]\, \mathbb{P}_p[C]}{\mathbb{P}_p[A \mid C]\, \mathbb{P}_p[C]}$$

independence

$$\geq \mathbb{P}_p[w_e = 1] \quad \text{by assumption.}$$

(c). we need to show $q_i \geq p$

$\Leftrightarrow$

$$\mathbb{P}^*[w_{e_i} = 1 \mid A,\ w_{e_{[i-1]}} = \eta_{e_{[i-1]}}] \geq \mathbb{P}_p[w_{e_i} = 1]$$

LHS =

$$\sum_{x \in \{0,1\}^{i-1}} \mathbb{P}^*[\eta_{e_{[i-1]}} = x] \cdot \mathbb{P}^*[w_{e_i} = 1 \mid A,\ w_{e_{[i-1]}} = x]$$

$$\geq \sum_{x \in \{0,1\}^{i-1}} \mathbb{P}^*[\eta_{e_{[i-1]}} = x] \cdot \mathbb{P}^*[w_{e_i} = 1]$$

part (b), as $\{w_{e_{[i-1]}} = x\}$ is indep of edge e

$$= \mathbb{P}^*[w_{e_i} = 1].$$

we have that $\eta \geq \omega$

$\Rightarrow$ since A is increasing,

$$\mathbb{P}^*[\eta \in B] \geqslant \mathbb{P}^*[\omega \in B]$$

and LHS $= \mathbb{P}^*[\omega \in B \mid \omega \in A]$

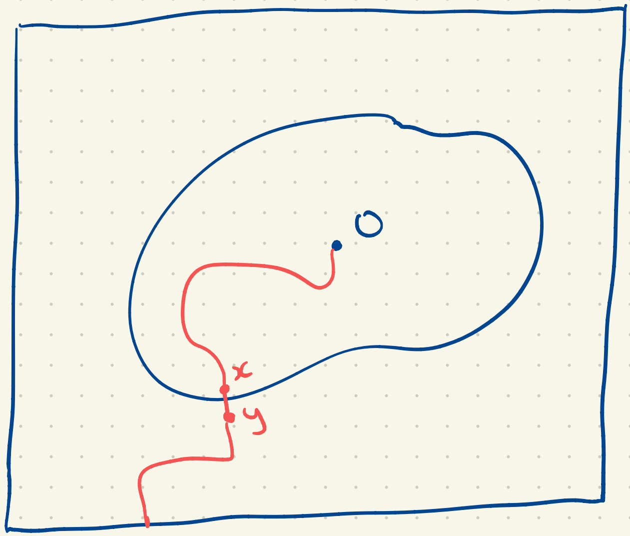
so $\mathbb{P}_p[B \mid A] \geqslant \mathbb{P}_p[B]$.

$\square$

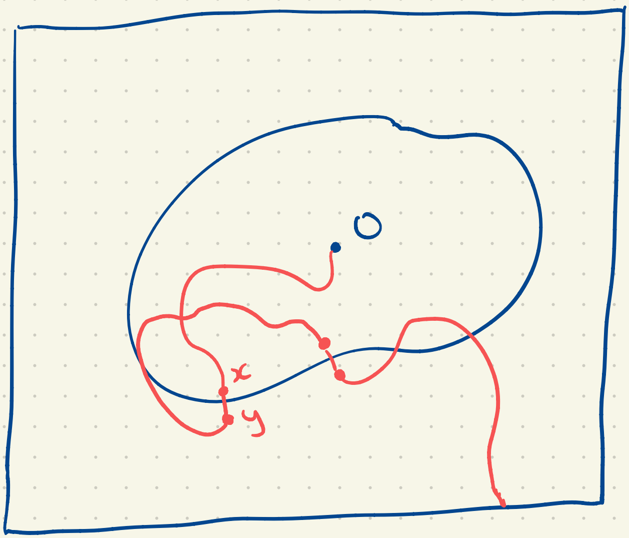
③ we have $\{0 \leftrightarrow \partial\Lambda_{kL}\} =$

$$\bigcup_{xy \in \Delta S} \{0 \overset{S}{\leftrightarrow} x,\ \omega_{xy} = 1\} \circ \{y \leftrightarrow \partial\Lambda_{kL}\}$$

indeed,



and if the path $y \leftrightarrow \partial\Lambda_n$ connects to the path $0 \overset{S}{\leftrightarrow} x$, we get a new xy:



so $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{kL}]$

$$\underset{\text{union bound}}{\leq} \sum_{xy \in \Delta S} \mathbb{P}_p\left[\{0 \overset{S}{\leftrightarrow} x,\ \omega_{xy}=1\} \circ \{y \leftrightarrow \partial\Lambda_{kL}\}\right]$$

$$\underset{\text{BK-Reimer}}{\leq} \sum_{xy \in \Delta S} \mathbb{P}_p\left[\{0 \overset{S}{\leftrightarrow} x\} \cap \{\omega_{xy}=1\}\right] \cdot \mathbb{P}_p\left[y \leftrightarrow \partial\Lambda_{kL}\right]$$

as $\{0 \overset{S}{\leftrightarrow} x\}$ and $\{\omega_{xy}=1\}$ are dependent on disjoint sets of edges,

$$\{0 \overset{S}{\leftrightarrow} x\} \cap \{\omega_{xy}=1\} = \{0 \overset{S}{\leftrightarrow} x\} \circ \{\omega_{xy}=1\} \ ;$$

we now apply BK-Reimer again:

$$\leq \sum_{cy \in \Delta S} \mathbb{P}_p\left[0 \overset{S}{\leftrightarrow} x\right] \mathbb{P}_p\left[\omega_{xy}=1\right] \mathbb{P}_p\left[y \leftrightarrow \partial\Lambda_{kL}\right]$$

and the proof proceeds from here as in the lecture notes.