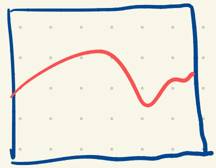
① • let $C_n =$  Λ_n ,

• $U_{n,k} = \{ \leq 1$ crossing from Λ_k to $\partial \Lambda_n \}$ where $k < n$. we know $\mathbb{P}_p[U_{n,k}] \to 1$ as $n \to \infty$ for all p.

• if $\theta(p) > 0$ then also

$$\lim_{k \to \infty} \mathbb{P}_p[\Lambda_k \leftrightarrow \infty] \to 1 \text{ as } k \to \infty.$$

• let $\varepsilon > 0$ & let k large enough s.t.

$$\mathbb{P}_p[\Lambda_k \leftrightarrow \infty] > 1-\varepsilon. \qquad ①$$

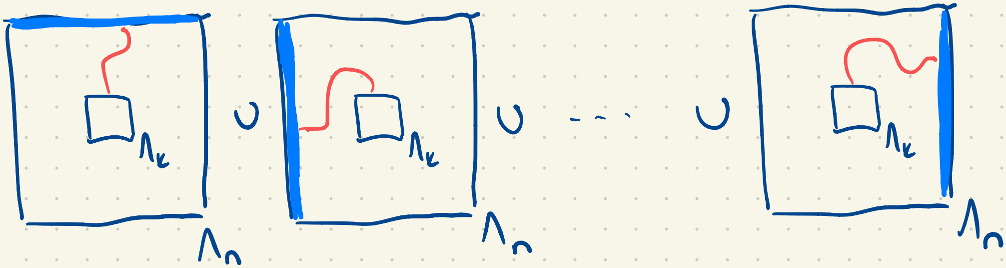
and then n large enough s.t.

$$\mathbb{P}_p[U_{n,k}] > 1-\varepsilon \qquad ②$$

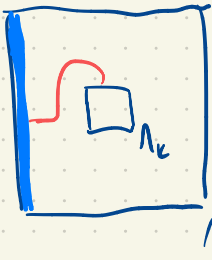
by ①, $\mathbb{P}_p[\Lambda_k \leftrightarrow \partial \Lambda_n] > 1-\varepsilon$

now

$\{\Lambda_k \leftrightarrow \partial\Lambda_n\} =$

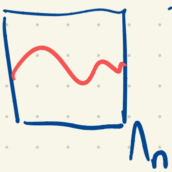


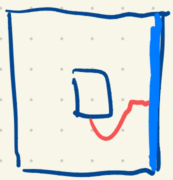
so by the square root trick,

$\mathbb{P}_p\Big[$ 

$\Big] \geq 1-\varepsilon^{\frac{1}{2d}}$

now

$\mathbb{P}_p\Big[$ 

$\Big] \geq \mathbb{P}_p\Big[$  $\cap$  $\cap\, U_{k,n}\Big]$

$\geq 1-2\varepsilon-\varepsilon^{\frac{1}{2d}}$

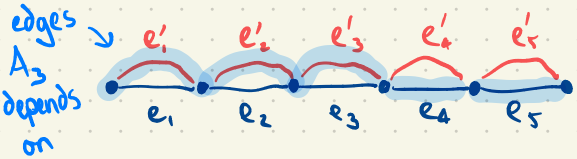
as $n \to \infty$.

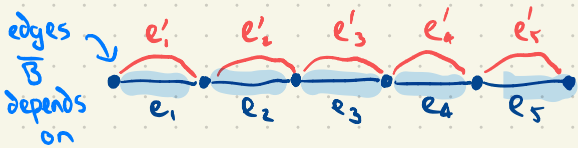
as ε is arbitrary, this shows $\mathbb{P}_p[C_n] \to 1$

② recall $w^{(i)} = (w_1', w_2', \dots, w_i', w_{i+1}, \dots, w_n)$.

note $w^{(0)} = w$, $w^{(n)} = w'$.

- let $\overline{A}_i = \{\overline{w} : w^{(i)} \in A\}$, $\overline{B} = \{\overline{w} : w \in B\}$





we want: $\boxed{*}$ $\overline{\mathbb{P}}_p[\overline{A}_{i-1} \circ \overline{B}] \le \overline{\mathbb{P}}_p[\overline{A}_i \circ \overline{B}]$

- note the difference between $\overline{A}_{i-1}$ and $\overline{A}_i$ is that $\overline{A}_{i-1}$ is dep. on e_i, whereas $\overline{A}_i$ is dep. on e_i'.

$\boxed{1}$ s well-defined

- let $\overline{w} \in \overline{A}_{i-1} \circ \overline{B}$ st. it has witnesses I, J st. $e_i \notin I$. then $s(\overline{w}) = \overline{w} \in \overline{A}_i \circ \overline{B}$ as I is also a witness of $\overline{A}_i$ for $\overline{w}$.

- let $\overline{w} \in \overline{A}_{i-1} \circ \overline{B}$ st. all witnesses I, J have $e_i \in I$. let

$$s(\overline{w})(e) = \begin{cases} \overline{w}(e) & e \neq e_i, e_i' \\ \overline{w}(e_i') & e = e_i \\ \overline{w}(e_i) & e = e_i' \end{cases}$$

then defining I' as I but with e_i' replacing e_i,
we see $I' \cap J$ and I' is a witness of $\bar{A}_i$ for $s(\bar{\omega})$.

so s well-defined.

$\boxed{2}$ s injective

s is clearly injective on $\{\bar{\omega} : e_i \in I\} =: X$
and on $\{\bar{\omega} : e_i \notin I\} =: Y$
(the cases we dealt with above). Moreover,
if $\bar{\omega} \in X$, $\bar{\eta} \in Y$
then every witness I of $\bar{A}_i$ for $s(\bar{\eta})$ contains e_i',
and there's a witness I of $\bar{A}_i$ for $s(\bar{\omega})$
contained in $\{e_1', \ldots, e_{i-1}', e_{i+1}, \ldots, e_n\}$, & so
does not contain e_i'. hence $s(\bar{\eta}) \neq s(\bar{\omega})$.

$\boxed{3}$ s measure-preserving

since s is either the identity or swaps
two edges, it's clear that

$$o(\bar{\omega}) = o(s(\bar{\omega})), \quad c(\bar{\omega}) = c(s(\bar{\omega}))$$

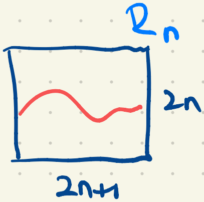
so $\bar{\mathbb{P}}_p[\bar{\omega}] = \bar{\mathbb{P}}_p[s(\bar{\omega})]$.

now $\overline{\mathbb{P}}_p\left[\overline{A}_{i-1} \circ \overline{B}\right] = \sum_{w \in \overline{A}_{i-1} \circ \overline{B}} \overline{\mathbb{P}}_p[w]$

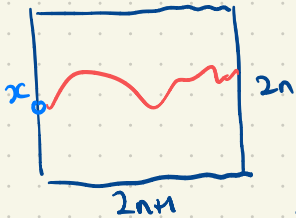
$$= \sum_{w \in \overline{A}_{i-1} \circ \overline{B}} \overline{\mathbb{P}}_p[s(\overline{w})]$$

$$\leq \sum_{\overline{\eta} \in \overline{A}_i \circ \overline{B}} \overline{\mathbb{P}}_p[\overline{\eta}] = \overline{\mathbb{P}}_p\left[\overline{A}_i \circ \overline{B}\right].$$

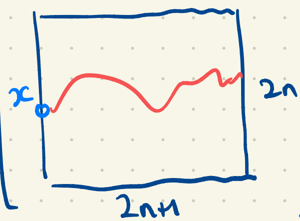
③ ⓐ $\frac{1}{2} = \mathbb{P}_{\frac{1}{2}}\left[H_n\right]$

$= \mathbb{P}_{\frac{1}{2}}\Big[$ 

$\Big]$

$= \mathbb{P}_{\frac{1}{2}}\Big[\bigcup_{\substack{x \in \text{LHS} \\ \text{of } R_n}}$ 

$\Big]$

union bound $\leq \sum_{\substack{x \in \\ \text{LHS of } R_n}} \mathbb{P}_{\frac{1}{2}}\Big[$ 

$\Big]$

if $x \leftrightarrow$ RHS of R_n, then

$x \leftrightarrow \partial\Lambda_{2n}(x)$

so above $\leq \sum_{\substack{x \in \\ \text{LHS of } R_n}} \mathbb{P}_{\frac{1}{2}}\left[x \leftrightarrow \partial\Lambda_{2n}(x)\right]$

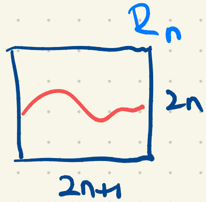
translation invariance.

$= 2n \, \mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_{2n}\right]$

so $\theta_{2n}\left(\frac{1}{2}\right) \geq \frac{1}{4n}$

(b) $\frac{1}{2} = \mathbb{P}_{\frac{1}{2}}\left[H_n\right]$

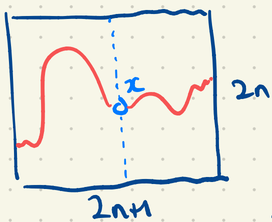
$= \mathbb{P}_{\frac{1}{2}}\Big[$



$\Big]$

$\{x \leftrightarrow \text{LHS of } R_n\} \circ \{x \leftrightarrow \text{RHS of } R_n\}$

$= \mathbb{P}_{\frac{1}{2}}\Big[\bigcup_{\substack{x \text{ on} \\ \text{centre of} \\ R_n}}$



$\Big]$

formally, $x \in \{(0,y) : -n \leq y \leq n\}$

then if $\{x \leftrightarrow \text{LHS of } R_n\} \circ \{x \leftrightarrow \text{RHS of } R_n\}$ occurs,

then $\{x \leftrightarrow \partial\Lambda_n(x)\} \circ \{x \leftrightarrow \partial\Lambda_n(x)\}$ occurs,

so above is

$$\leq \sum_{\substack{x \in \text{centre} \\ \text{line of } R_n}} \mathbb{P}_{\frac{1}{2}}\left[\{x \leftrightarrow \partial\Lambda_n(x)\} \circ \{x \leftrightarrow \partial\Lambda_n(x)\}\right]$$

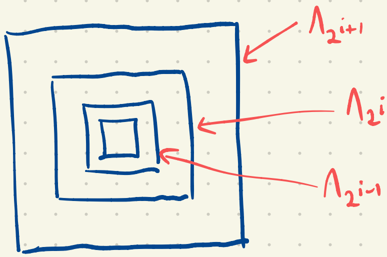
$$= 2n \; \mathbb{P}_{\frac{1}{2}}\left[\{0 \leftrightarrow \partial\Lambda_n\} \circ \{0 \leftrightarrow \partial\Lambda_n\}\right]$$

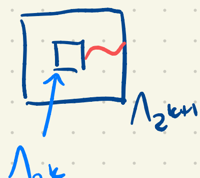
so by BK-Reimer,

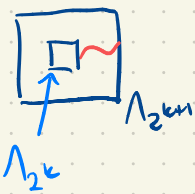
$$\mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_n\right] \geq \mathbb{P}_{\frac{1}{2}}\left[\{0 \leftrightarrow \partial\Lambda_n\} \circ \{0 \leftrightarrow \partial\Lambda_n\}\right]^{1/2}$$

$$\geq \left(\frac{1}{4n}\right)^{1/2} = \frac{1}{2n^{1/2}}$$

(c) consider nested annuli



then $\mathbb{P}_{\frac{1}{2}}\left[0 \longleftrightarrow \partial\Lambda_{2^i}\right] \leq \mathbb{P}_{\frac{1}{2}}\Big[\bigcap_{k=0}^{i-1}$  $\Big]$

independence
$= \prod_{k=0}^{i-1} \mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] \leq (1-c)^i$

doing a change of variables $n = 2^i$ gives

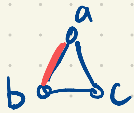
$$\mathbb{P}_{\frac{1}{2}}\left[0 \longleftrightarrow \partial\Lambda_n\right] \leq n^{-c'} \quad \text{for some } c' > 0$$

and extending to $2^i < n < 2^{i+1}$ is straightforward.

4. (a) we want $\mathbb{P}_p[u \leftrightarrow v] = \mathbb{P}'_{1-p}[u \leftrightarrow v]$ for all $u, v \in G$.

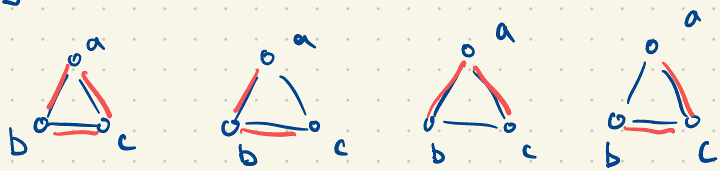
since $\mathbb{P}_p$ and $\mathbb{P}'_{1-p}$ are identical outside of the triangle abc, we focus just on the triangle.

$\mathbb{P}_p[a \leftrightarrow b \not\leftrightarrow c] = p(1-p)^2$

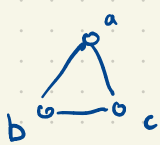


& same for $\mathbb{P}_p[a \leftrightarrow c \not\leftrightarrow b]$, $\mathbb{P}_p[a \not\leftrightarrow b \leftrightarrow c]$.

$\mathbb{P}_p[a \leftrightarrow b \leftrightarrow c] = p^3 + 3p^2(1-p)$



$\mathbb{P}_p[a \not\leftrightarrow b \not\leftrightarrow c,\ a \not\leftrightarrow c] = (1-p)^3$



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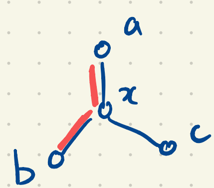
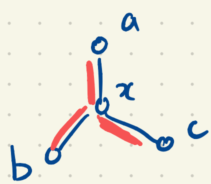
in $\mathbb{P}'_{1-p}$, we have

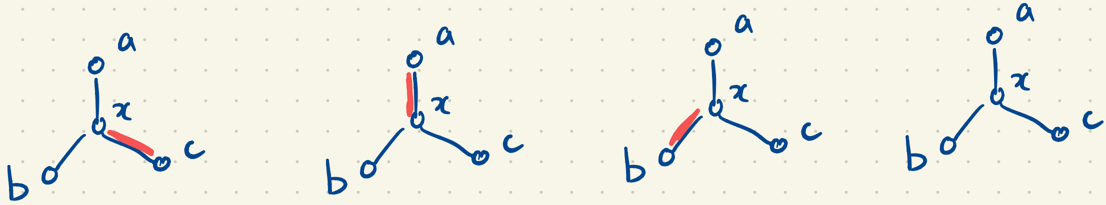
$\mathbb{P}'_{1-p}[a \leftrightarrow b \not\leftrightarrow c] = (1-p)^2 p$



$\mathbb{P}'_{1-p}[a \leftrightarrow b \leftrightarrow c] =$



$\mathbb{P}'_{1-p}[a \not\leftrightarrow b \not\leftrightarrow c,\ a \not\leftrightarrow c] = p^3 + 3p^2(1-p)$



note that $\mathbb{P}_p[\text{connection}] = \mathbb{P}'_{1-p}[\text{connection}]$

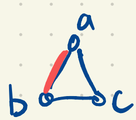
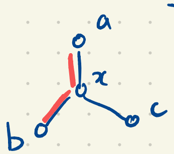
for each connection above holds iff

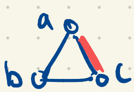
$$p^3 + 3p^2(1-p) = (1-p)^3$$

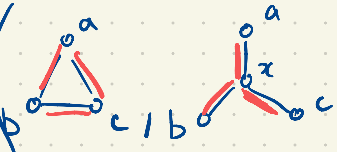
$$\Leftrightarrow p^3 - 3p^3 + 3p^2 + p^3 - 3p^2 + 3p - 1 = 0$$

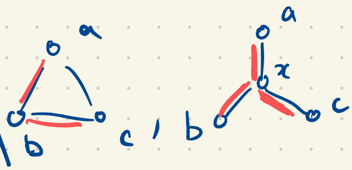
$$\Leftrightarrow -p^3 + 3p - 1 = 0 .$$

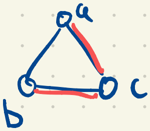
the coupling $\mathbb{P}^*$ on $\{0,1\}^{E(abc)} \times \{0,1\}^{E(abcx)}$

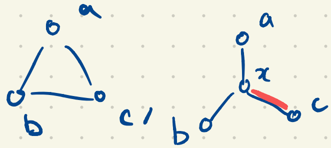
is: $\mathbb{P}^*\Big[\Big($  ,  $\Big)\Big] = p(1-p)^2$

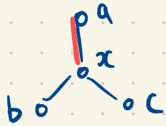
& similar for  , etc,

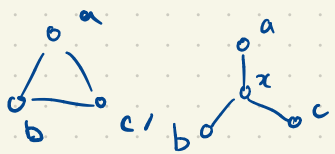
$\mathbb{P}^*\Big[\Big($  $\Big)\Big] = p^3$

$\mathbb{P}^*\Big[\Big($  $\Big)\Big] = p^2(1-p)$

& similar for  , etc ,

$\mathbb{P}^*\Big[\Big($  $\Big)\Big] = p^2(1-p)$

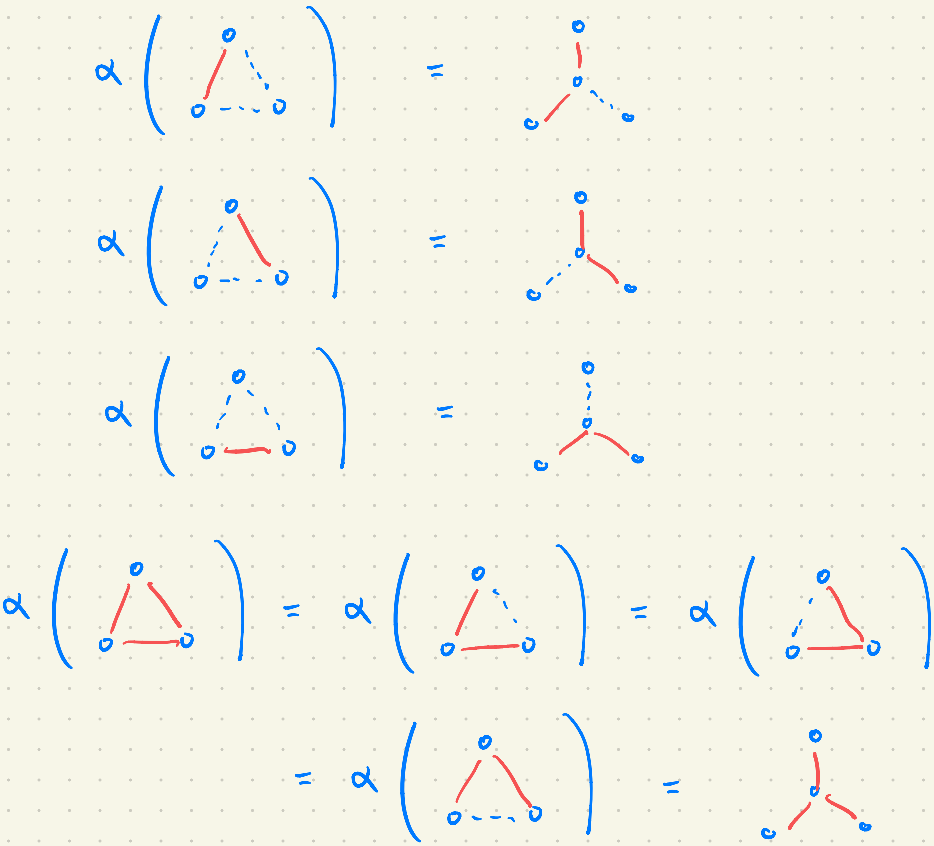
& similar for 

$\mathbb{P}^*\Big[\Big($  $\Big)\Big] = p^3$

and $\mathbb{P}^*\big[(\omega, \omega')\big] = 0$ otherwise.

one can find that the marginals of $\mathbb{P}^*$ on $\{0,1\}^{E(abc)}$ and $\{0,1\}^{E(abcx)}$ are $\mathbb{P}_p$, $\mathbb{P}'_{1-p}$ resp.

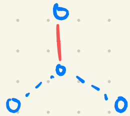
alternatively, one can define the coupling via a (random) map $\alpha : \{0,1\}^{E(abc)} \to \{0,1\}^{E(abcx)}$ which preserves connections. let's define α:

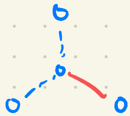


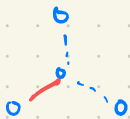
for the remaining case  (no open edges), α is random, in the sense that:

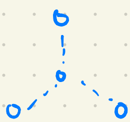
if $\omega =$ , we set

$\alpha(\omega) =$

 with probability $\dfrac{p^2(1-p)}{(1-p)^3}$

 w/ prob $\dfrac{p^2(1-p)}{(1-p)^3}$

 w/ prob $\dfrac{p^2(1-p)}{(1-p)^3}$

 w/ prob $\dfrac{p^3}{(1-p)^3}$

by above, these probabilities sum to 1.

then we set $\mathbb{P}^* : \{0,1\}^{E(abc)} \times \{0,1\}^{E(abcx)} \to [0,1]$

as $\mathbb{P}^*[(\omega,\eta)] = \mathbb{P}_p[\omega] \cdot \mathbb{P}[\alpha(\omega) = \eta]$.

this gives the same $\mathbb{P}^*$ as above

and moreover $\mathbb{P}^*[(\omega,\eta)] > 0$ iff ω, η have same connection properties on vertices of abc.

one can extend this to a coupling on $\{0,1\}^{E(G)} \times \{0,1\}^{E(G')}$ easily.

let $\mathbb{P}^*_{GG'}[(\omega, \omega')]$

$:= \mathbb{1}\{\omega = \omega' \text{ on edges outside } abc\}$

$\cdot \mathbb{P}_p\big|_{E \setminus abc}[\omega|_{E \setminus abc}]$

$\cdot \mathbb{P}^*[(\omega|_{abc}, \omega'|_{abcx})]$

where $\mathbb{P}^*$ is the coupling above.

one can check the conditions on the marginals again.

(b) starting with $\mathbb{T}$ and applying star-triangle to every triangle in $\mathbb{T}$ gives $\mathbb{H}$. the measure $\mathbb{P}_p^{\mathbb{T}}$ becomes $\mathbb{P}_{1-p}^{\mathbb{H}}$ in this process, when $p^3+1=3p$, and preserves connection probabilities.

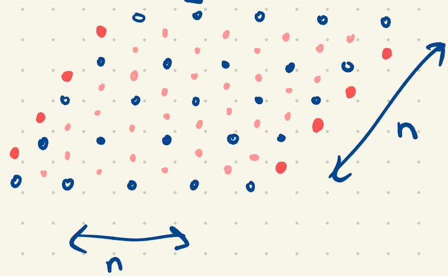
to be a bit more rigorous, our approach above lets us couple $\mathbb{P}_p^{\mathbb{T}}$ with the measure $\mathbb{P}'$ obtained by applying star-triangle to all triangles in B_n

(distance $\leq n$ from origin). then $\mathbb{P}'|_{B_n}$ is $\mathbb{P}^{\mathbb{H}}_{1-p}$ restricted to B_n. moreover we have

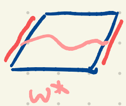
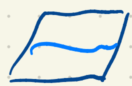
$$\mathbb{P}^{\mathbb{T}}_p[0 \leftrightarrow \partial B_n] = \mathbb{P}'[0 \leftrightarrow \partial B_n] = \mathbb{P}^{\mathbb{H}}_{1-p}[0 \leftrightarrow \partial B_n]$$

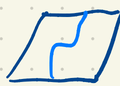
when $p^3+1 = 3p$. let $n \to \infty$.

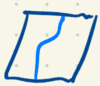
(c) first, notice that if $\omega \sim \mathbb{P}^{\mathbb{T}}_p$, then $\omega^* \sim \mathbb{P}^{\mathbb{H}}_{1-p}$, ω^* the dual configuration.

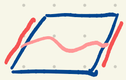
let $\Lambda_n =$  in $\mathbb{T}$.

at $p = p_0$ satisfying $p_0^3+1 = 3p_0$,

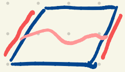
$\mathbb{P}^{\mathbb{T}}_p\Big[$  $\Big] = 1 - \mathbb{P}^{\mathbb{T}}_p\Big[$  $\Big] \leq 1 - \mathbb{P}^{\mathbb{H}}_{1-p}\Big[$  $\Big]$

by part (b) and symmetry $= 1 - \mathbb{P}^{\mathbb{T}}_p\Big[$  $\Big]$

$\Rightarrow \mathbb{P}^{\mathbb{T}}_p\Big[$  $\Big] \leq \frac{1}{2}$

and $\mathbb{P}^{\mathbb{H}}_{1-p}\Big[$  $\Big] \geq \frac{1}{2}$.

now similar to our proof on $\mathbb{Z}^2$,

$\mathbb{P}^{\mathbb{H}}_{1-p}\Big[$  $\Big] \geq \frac{1}{2}$ & exponential decay at $1-p$ for $\mathbb{P}^{\mathbb{H}}_{1-p}$ are incompatible.

so $1-p_0 \geq p_c(\mathbb{H})$

a similar argument reversing the roles of $\mathbb{T}$ and $\mathbb{H}$ gives

$$\mathbb{P}^{\mathbb{T}}_{p}[\text{some crossing event}] \geq \frac{1}{2} \quad \text{at } p = p_0$$

which is incompatible with exponential decay, so

$$p_0 \geq p_c(\mathbb{T})$$

$\boxed{*}$ so $p_c(\mathbb{T}) \leq p_0 \leq 1 - p_c(\mathbb{H}) \Rightarrow p_c(\mathbb{T}) + p_c(\mathbb{H}) \leq 1.$

- to prove $p_c(\mathbb{T}) + p_c(\mathbb{H}) \geq 1$, we redo Zhang's argument.

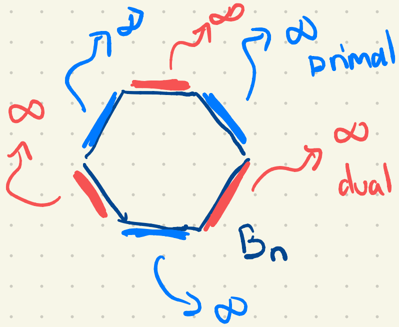
if $p_c(\mathbb{T}) + p_c(\mathbb{H}) < 1$, then $\exists\ p$ s.t.

$$\mathbb{P}^{\mathbb{T}}_{p}[\infty \text{ cluster}] = 1$$

$$\mathbb{P}^{\mathbb{H}}_{1-p}[\infty \text{ cluster}] = 1$$

using the square root trick, one finds, for example

$\mathbb{P}_p$ [



] $\geq 1-6a^{1/6}$

$$\Rightarrow \mathbb{P}_p\left[\geq 2 \ \infty \text{ clusters in primal or dual}\right] \geq 1-6a^{1/6}$$

& this contradicts uniqueness of the infinite cluster.

∎

(d) $\boxed{*}$ and the second part of (c) give $p_c(\mathbb{T}) = p_0$

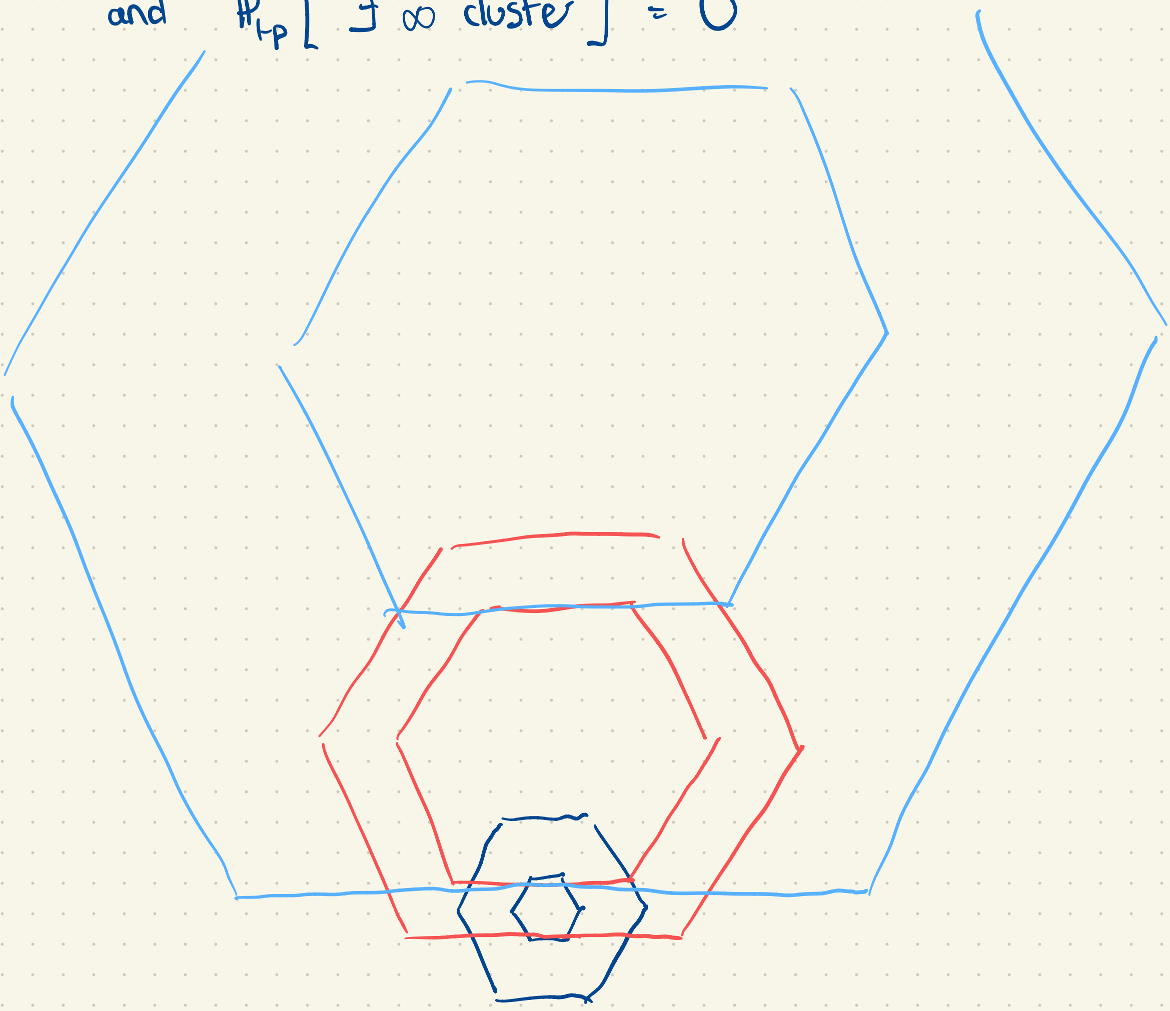
alternative proof of 1st part of (c) :

(by osama) $\qquad p_c(\mathbb{T}) + p_c(\mathbb{H}) \le 1.$

- assume $p_c(\mathbb{T}) + p_c(\mathbb{H}) > 1.$ then $\exists\, p$ s.t.

$$\mathbb{P}_p^{\mathbb{T}}\left[0 \leftrightarrow \partial\Lambda_n\right] \le e^{-cn}$$

and $\mathbb{P}_{1-p}^{\mathbb{H}}\left[\exists\, \infty \text{ cluster}\right] = 0$



consider a sequence of annuli nested as shown, of sizes:

$$\Lambda_{2n_0} \setminus \Lambda_{n_0}, \quad \Lambda_{2\lambda n_0} \setminus \Lambda_{(2\lambda-1)n_0},$$

$$\Lambda_{2\lambda^2 n_0} \setminus \Lambda_{(2\lambda^2-2\lambda+1)n_0}, \quad \ldots,$$

$$\Lambda_{2\lambda^k n_0} \setminus \Lambda_{(2\lambda^k - 2\lambda^{k-1} + 2\lambda^{k-2} - \cdots)n_0}.$$

let A_k be the event of a circuit in the dual ω^* around the k^{th} annulus.

then $\mathbb{P}_p^\pi[\exists \infty \text{ dual cluster in } \omega^*]$

$$\geq \mathbb{P}_p^\pi\left[\bigcap_{i=1}^{\infty} A_i\right]$$

$$= \lim_{k\to\infty} \mathbb{P}_p^\pi\left[\bigcap_{i=1}^{k} A_i\right]$$

$$\overset{\text{FKG}}{\geq} \lim_{k\to\infty} \prod_{i=1}^{k} \mathbb{P}_p^\pi[A_i]$$

$$= \lim_{k\to\infty} \prod_{i=1}^{k} 1 - \mathbb{P}_p^\pi\left[\Lambda_{(2\lambda^i\cdots)n_0} \leftrightarrow \partial\Lambda_{2\lambda^i n_0}\right]$$

now this $\mathbb{P}_p^{\pi}\left[\Lambda_{(2\lambda^i \cdots)n_0} \leftrightarrow \partial \Lambda_{2\lambda^i n_0}\right]$

$$\leq |\Lambda_{(2\lambda^i \cdots)n_0}| \, e^{-c(2\lambda^{i-1} - \cdots)n_0}$$

$$= K \cdot (2\lambda^i \cdots) n_0 \, e^{-c(2\lambda^{i-1} - \cdots)n_0}$$

$$\leq K' \lambda^i e^{-c' \lambda^{i-1}} \qquad \text{for some } K', c'$$

now $\prod_{i=1}^{\infty} \left(1 - K' \lambda^i e^{-c' \lambda^{i-1}}\right) > 0$

& this contradicts $\mathbb{P}_{1-p}^{H}[\exists \infty \text{ cluster}] = 0$.

$\blacksquare$