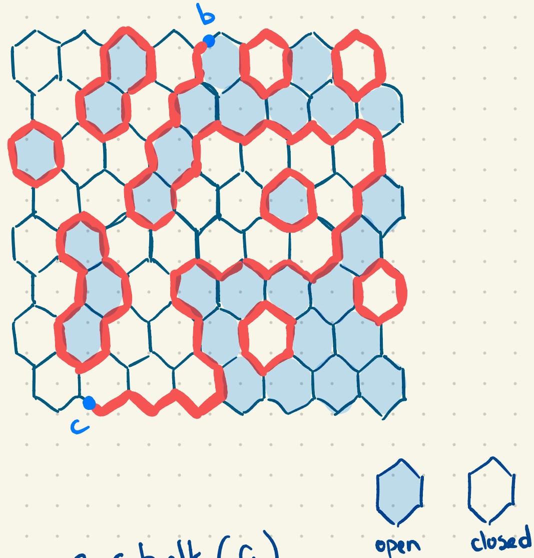
1 (a)



- let $\phi : \Omega_G \to \Omega^{\text{loop}}_{G,bc}$

as $e \in \phi(\omega)$

$$\text{iff} \begin{cases} \omega \text{ differs either side of } e & e \in \text{bulk}(G) \\ \omega = 1 \text{ on the ! face by } e & e \in (b,c) \subset \partial G \\ \omega = 0 \text{ on the ! face by } e & e \in (c,b) \subset \partial G \end{cases}$$

- $\phi(\omega)$ has a self-avoiding path $b \leftrightarrow c$ given by Γ = interface between open clusters touching (c,b) and closed clusters touching (b,c)

indeed, any path from (b,c) to (c,b) crosses an odd number of edges $e \in \phi(\omega)$ by defn of ϕ

(b) if γ_0 disconnects a from z, then there's no self-avoiding path $a \leftrightarrow z$ which is disjoint from γ_0, so in this case,

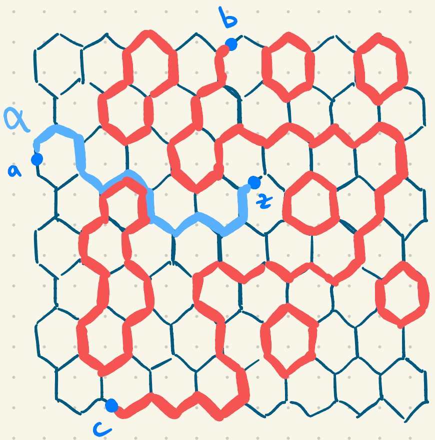
$$\Omega^{\text{loop}}_{G,az,bc}[\gamma_0] = \emptyset.$$

in the case that γ_0 does not disconnect a from z, let α be some fixed path $a \leftrightarrow z$ avoiding γ_0. then the map

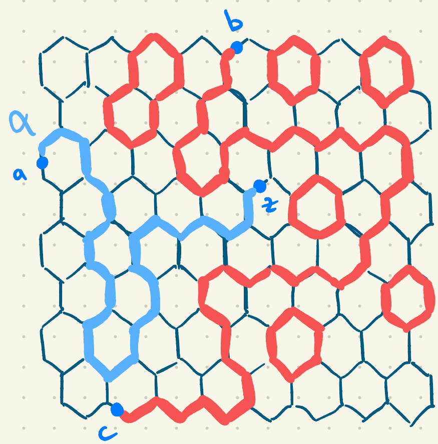
$$\psi : \Omega^{\text{loop}}_{G,bc}[\gamma_0] \to \Omega^{\text{loop}}_{G,az,bc}[\gamma_0]$$

given by $\psi(\eta) = \eta \Delta \alpha$ is a bijection with inverse $\psi^{-1} = \psi$.

symmetric difference



$\xrightarrow{\psi}$



(c) using the above,

$$F_a(z) = \frac{|\Omega^{\text{loop}}_{G,az,bc}|}{|\Omega_G|}$$

part (b)
$$= \frac{1}{|\Omega_G|} \sum_{\substack{\eta_0 : b \leftrightarrow c \\ \text{doesn't disconnect} \\ a \text{ and } z}} |\Omega^{\text{loop}}_{G,bc}[\eta_0]|$$

part (a)
$$= \frac{1}{|\Omega_G|} \sum_{\substack{\eta_0 : b \leftrightarrow c \\ \text{doesn't disconnect} \\ a \text{ and } z}} |\{\omega \in \Omega_G : \Gamma(\omega) = \eta_0\}|$$

$$= \frac{1}{|\Omega_G|} |\{\omega \in \Omega_G : \Gamma(\omega) \text{ doesn't disconnect } a \text{ and } z\}|$$

$$= \mathbb{P}_{\frac{1}{2}}[\Gamma(\omega) \text{ doesn't disconnect } a \text{ and } z]$$

(c) by above,

$$|F_a(z) - F_a(z')| = \Big|\mathbb{P}_{\frac{1}{2}}[\Gamma \text{ disconnects } z \text{ from } a]$$

$$- \mathbb{P}_{\frac{1}{2}}[\Gamma \text{ disconnects } z \text{ from } a]\Big|$$

for sets A, B, $\big||A| - |B|\big| = \big||A \setminus B| - |B \setminus A|\big|$

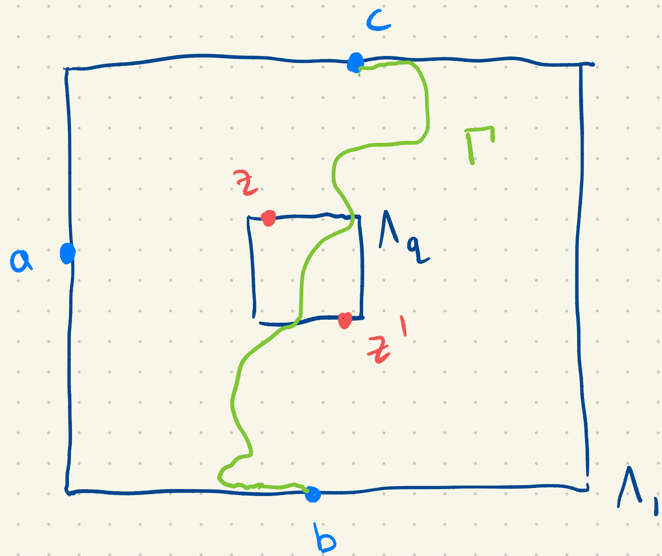
$$\leq |(A \setminus B) \cup (B \setminus A)|$$

$$= |A \triangle B|$$

so above is $\leq \mathbb{P}_{\frac{1}{2}}[\Gamma$ disconnects exactly one of z, z'

from $a]$

(e)



Γ separates exactly one of z, z' from a

$\Rightarrow$ Γ intersects Λ_q

$\Rightarrow$ $\exists$ open cluster $\Lambda_q \leftrightarrow \Lambda_1$

$$\Rightarrow \; |F_a(z) - F_a(z')| \le c \cdot q^{\alpha} \le c \cdot |z - z'|^{\alpha}.$$

② (a) • colour the faces of $\mathbb{Z}^2$ black or white according to a chessboard pattern, ie: face f is black iff its lower left corner (x,y) has $x+y$ even.

• then the black faces form a copy of $\mathbb{Z}^2$, where the black faces are the vertices & are connected by an edge iff they share a corner. call this graph $\mathbb{L}$.

• the white faces similarly are a copy of $\mathbb{Z}^2$, and in fact it's the dual of $\mathbb{L}$, $\mathbb{L}^*$.

• a mirror is an edge of $\mathbb{L}$ if it joins two black faces & an edge of $\mathbb{L}^*$ o/w.

• when $p=1$, every vertex of $\mathbb{Z}^2$ is assigned a mirror, with its orientation chosen independently & uniformly

so each vertex is assigned either an edge of $\mathbb{L}$ or an edge of $\mathbb{L}^*$.

• let w = the collection of edges of $\mathbb{L}$ assigned.

it is straightforward that $\omega \sim \mathbb{P}_{\frac{1}{2}}$, and so the edges of $\mathbb{L}^*$ assigned are ω^*, with $\omega^* \sim \mathbb{P}_{\frac{1}{2}}$ too.

- when $p=1$, the loops in the model of mirrors separate primal and dual clusters in ω, ω^*; that is, each loop (properly oriented) has a cluster of ω on its left and a cluster of ω^* on its right.

- $\exists$ ∞ loop, then $\exists$ ∞ primal cluster on its left & an ∞ dual cluster on its right.

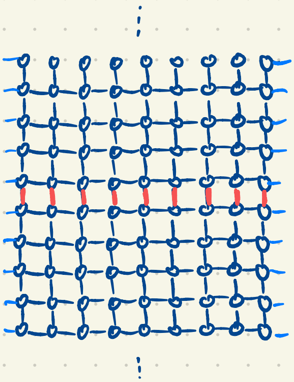
so $\mathbb{P}_1[0 \leftrightarrow \infty] \leq \mathbb{P}_{\frac{1}{2}}[\infty \text{ cluster in } \omega, \infty \text{ cluster in } \omega^*] = 0$

by zhang's argument

■

(b) consider the $2n+1$ vertical edges of $\mathbb{Z}_{2n+1} \times \mathbb{Z}$

$\{(x,0),(x,1) : 1 \le x \le 2n+1\}$. (shown in red below).



every finite loop (and ∞ path connecting $+\infty \leftrightarrow +\infty$ or $-\infty$ to $-\infty$) must cross an even number of these edges

$\Rightarrow$ the total number on finite loops (and these ∞ paths) must be even

$\Rightarrow$ $\exists$ at least one on an infinite path top $\leftrightarrow$ bottom.

(c) one of the red edges e_0 above is incident to 0.

by translation invariance, $\mathbb{P}[e \leftrightarrow \infty] = \mathbb{P}[e' \leftrightarrow \infty]$ for all e, e' red edges. so

$$1 = \mathbb{P}_p[\text{at least one } e \leftrightarrow \infty] = \mathbb{P}_p\Big[\bigcup_{e \text{ red}} e \leftrightarrow \infty\Big]$$

$$\underset{\text{union bound}}{\le} \sum_{e \text{ red}} \mathbb{P}_p[e_0 \leftrightarrow \infty] \le 2n+1 \; \mathbb{P}_p[0 \leftrightarrow \infty]$$

now $0 \leftrightarrow \infty \Rightarrow 0 \leftrightarrow \partial\Lambda_n$ so

$$\mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_n \text{ in } \mathbb{Z}_{2n+1} \times \mathbb{Z}\right] \geq \frac{1}{2n+1}$$

(d) the event $0 \leftrightarrow \partial\Lambda_n$ is dependent only on the configuration strictly inside Λ_n. since the config inside is independent of outside, we have that it is distributed the same as that inside Λ_n on $\mathbb{Z}^2$.

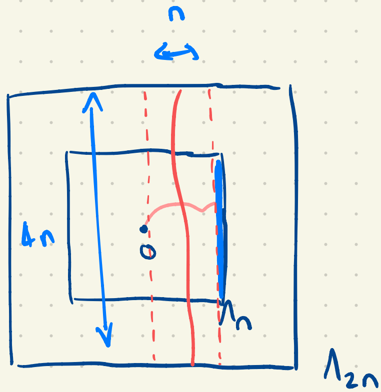
hence

$$\mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_n \text{ in } \mathbb{Z}_{2n+1} \times \mathbb{Z}\right]$$

$$= \mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_n \text{ in } \mathbb{Z}^2\right]$$

$$\geq \frac{1}{2n+1}.$$

3. (a)



it is clear that $\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \text{right side of } \partial\Lambda_n]$

$$\geq \frac{1}{4}\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n]$$

let A_n be the event that $[0,n]\times\{-2n\} \leftrightarrow [0,n]\times\{2n\}$ in the box $[0,n]\times[-2n,2n]$ (seen in red above) then by RSW, $\exists c>0$ st. $\forall n\geq 1$

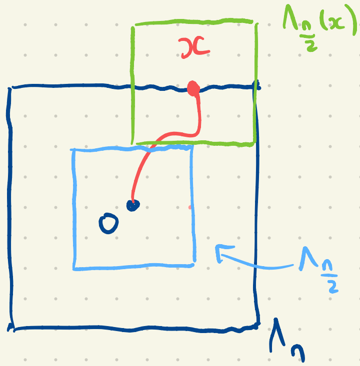
$$\mathbb{P}_{\frac{1}{2}}[A_n] \geq c.$$

now $\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \Lambda_{2n}] \geq \mathbb{P}_{\frac{1}{2}}[A_n \cap \{0 \leftrightarrow \text{right side of } \partial\Lambda_n\}]$

$$\underset{\text{FKG}}{\geq} \mathbb{P}_{\frac{1}{2}}[A]\,\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \text{right side of } \partial\Lambda_n]$$

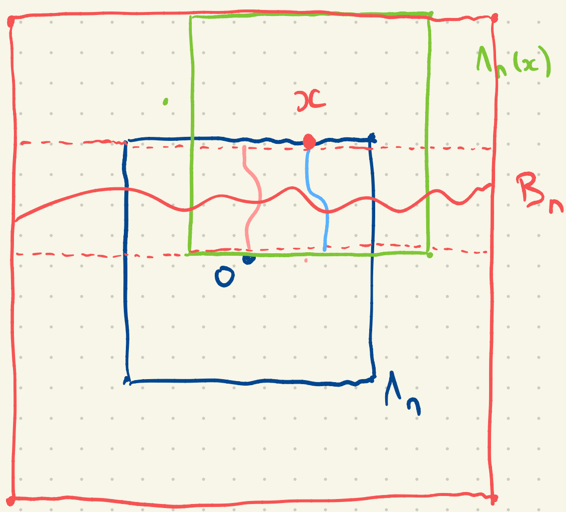
$$\geq \frac{c}{4}\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \Lambda_n].$$

(b)



$$\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow x] \leq \mathbb{P}_{\frac{1}{2}}\left[\{0 \leftrightarrow \partial\Lambda_{\frac{n}{2}}\} \cap \{x \leftrightarrow \partial\Lambda_{\frac{n}{2}}(x)\}\right]$$

$$\underset{\text{independence}}{=} \mathbb{P}_{\frac{1}{2}}\left[\{0 \leftrightarrow \partial\Lambda_{\frac{n}{2}}\}\right]^2 \underset{\text{part (a)}}{\leq} C \cdot \mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_n\right]^2$$



wlog assume $x \in$ top of $\partial\Lambda_n$.

$$\{0 \leftrightarrow \mathrm{top}(\partial\Lambda_n)\} \cap \{x \leftrightarrow \mathrm{bottom}(\partial\Lambda_n(x))\} \cap B_n \subset \{0 \leftrightarrow x\}$$

where $B_n = \{-2n\} \times [0,n] \leftrightarrow \{2n\} \times [0,n]$ in $[-2n, 2n] \times [0,n]$ (shown in red).

then $\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow x]$

$$\geq \mathbb{P}_{\frac{1}{2}}\left[\{0 \leftrightarrow \text{top}(\partial\Lambda_n)\} \cap \{x \leftrightarrow \text{bottom}(\partial\Lambda_n(x))\} \cap B_n\right]$$

$$\underset{\text{FKG}}{\geq} \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \text{top}(\partial\Lambda_n)]\,\mathbb{P}_{\frac{1}{2}}[x \leftrightarrow \text{bottom}(\partial\Lambda_n(x))]\,\mathbb{P}_{\frac{1}{2}}[B_n]$$

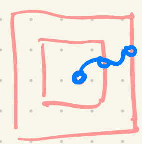
$$\underset{\text{RSW}}{\geq} \frac{1}{16}\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n]^2 \cdot c.$$

$N \geq 2n$

(c) let $1 \leq n \leq N$. show

$$\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow N] \leq \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n]\,\mathbb{P}_{\frac{1}{2}}[\partial\Lambda_n \leftrightarrow \partial\Lambda_N]$$

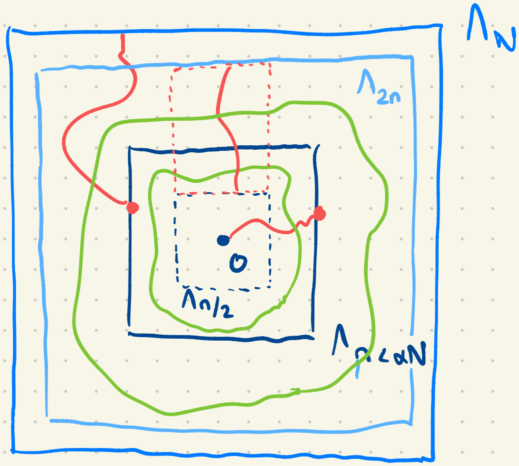
$$\leq c \cdot \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow N]$$



$$\{0 \leftrightarrow \partial\Lambda_N\} \subset \{0 \leftrightarrow \partial\Lambda_n\} \cap \{\partial\Lambda_n \leftrightarrow \partial\Lambda_N\}$$

well, $\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow N] \leq \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n,\ \partial\Lambda_n \leftrightarrow \partial\Lambda_N]$

$$\underset{\text{independence}}{=} \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n]\,\mathbb{P}_{\frac{1}{2}}[\partial\Lambda_n \leftrightarrow \partial\Lambda_N]$$



$$\mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_N\right] \geq$$

$$\mathbb{P}_{\frac{1}{2}}\Big[\{0 \leftrightarrow \partial\Lambda_n\} \cap \{\partial\Lambda_n \leftrightarrow \partial\Lambda_N\} \cap \{\text{circuit in } \Lambda_n \setminus \Lambda_{\frac{n}{2}}\}$$
$$\cap \{\text{circuit in } \Lambda_{2n} \setminus \Lambda_n\}$$
$$\cap \{\text{vertical crossing in } [-\tfrac{n}{2}, \tfrac{n}{2}] \times [\tfrac{n}{2}, \min\{2n, N\}]\}\Big]$$

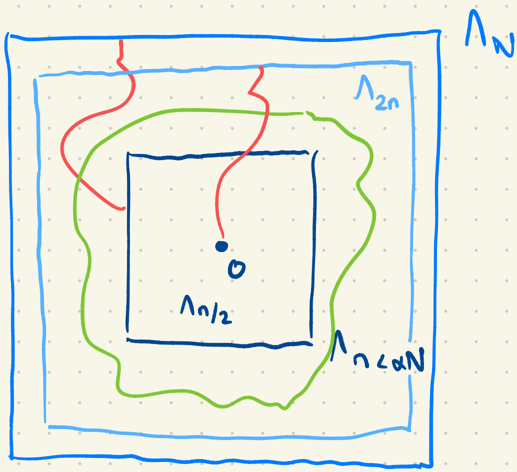
FKG

$$\geq \mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_n\right] \mathbb{P}_{\frac{1}{2}}\left[\partial\Lambda_n \leftrightarrow \partial\Lambda_N\right]$$
$$\mathbb{P}_{\frac{1}{2}}\left[\text{circuit in } \Lambda_n \setminus \Lambda_{\frac{n}{2}}\right]$$
$$\mathbb{P}_{\frac{1}{2}}\left[\text{circuit in } \Lambda_{2n} \setminus \Lambda_n\right]$$
$$\mathbb{P}_{\frac{1}{2}}\left[\text{vertical crossing in } [-\tfrac{n}{2}, \tfrac{n}{2}] \times [\tfrac{n}{2}, 2n]\right]$$

$$\geq \mathbb{P}_{\frac{1}{2}}\left[0 \leftrightarrow \partial\Lambda_n\right] \mathbb{P}_{\frac{1}{2}}\left[\partial\Lambda_n \leftrightarrow \partial\Lambda_N\right] \cdot c_1^2 \cdot c_2$$

RSW

shorter proof by tuomas



$$\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_N] \geq \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_{2n},\ \Lambda_n \leftrightarrow \partial\Lambda_N,\ \text{circuit in } \Lambda_{2n}\setminus\Lambda_n]$$

$$\overset{\text{FKG}}{\geq} \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_{2n}]\,\mathbb{P}_{\frac{1}{2}}[\Lambda_n \leftrightarrow \partial\Lambda_N]\,\mathbb{P}_{\frac{1}{2}}[\text{circuit in } \Lambda_{2n}\setminus\Lambda_n]$$

RSW and part (a)

$$\geq c_1 \cdot c_2 \cdot \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n]\,\mathbb{P}_{\frac{1}{2}}[\Lambda_n \leftrightarrow \partial\Lambda_N]$$