Percolation exercises 1

1. (a) Show that there is no phase transition in one dimension, that is: $p_c(\mathbb{Z}) = 1$.
   (b) Show that there is no phase transition on a strip, that is: $p_c(\mathbb{Z}\times\{0, 1, \ldots, n\}) = 1$.

2. (a) Given the definition of a sigma algebra $\mathbb{F}$ on a set Ω, show that: $\varnothing \in \mathbb{F}$, $\Omega \in \mathbb{F}$, and that $\mathbb{F}$ is closed under countable intersections.
   (b) Show that the events $\{x \leftrightarrow y\}$, $\{x \leftrightarrow \infty\}$, $\{\exists \text{ an } \infty \text{ cluster}\}$ all lie in the sigma algebra $\sigma(\mathbb{A})$, where $\mathbb{A}$ is the set of cylinder sets.

3. In this exercise, we prove the uniqueness part of Caratheodory's extension theorem. Let μ_1, μ_2 be two measures on $(\Omega, \mathbb{F})$, where $\mathbb{F}$ is a sigma algebra on Ω, with $\mu_1(\Omega) = \mu_2(\Omega) < \infty$. Let $\mu_1 = \mu_2$ on $\mathbb{A}$, where $\mathbb{A} \subset 2^{\Omega}$ satisfying $\varnothing \in \mathbb{A}$, and $A, B \in \mathbb{A} \Rightarrow A \cap B \in \mathbb{A}$, and $\sigma(\mathbb{A}) = \mathbb{F}$. We want to show that $\mu_1 = \mu_2$ on all of $\mathbb{F}$.
   (a) Let $\mathbb{D}$ be the set of sets A in $\mathbb{F}$ such that $\mu_1(A) = \mu_2(A)$. Show first that $\mathbb{D}$ is closed under intersections and complements.
   (b) Let $A_i \in \mathbb{D}$ and $A_i \subset A_{i+1}$ for all $i \in \mathbb{N}$. Show that $\bigcup_{i=1}^{\infty} A_i \in \mathbb{D}$.
   (c) Show that the above properties are enough to show that $\mathbb{D}$ is a sigma algebra. Conclude that $\mathbb{D} = \mathbb{F}$.

4. Prove the corollary to Caratheodory's extension theorem, that is, show that for all events A in our sigma-algebra $\mathbb{F} = \sigma(\mathbb{A})$, and for all $\varepsilon > 0$, there exists a cylinder set $A_\varepsilon \in \mathbb{A}$ such that

$$\mathbb{P}_p[A \triangle A_\varepsilon] \leq \varepsilon, \tag{1}$$

   where $A \triangle A_\varepsilon = (A \setminus A_\varepsilon) \cup (A_\varepsilon \setminus A)$, called the symmetric difference of A and A_ε. You can use the following identity: for all $A \in \mathbb{F} = \sigma(\mathbb{A})$,

$$\mu[A] = \inf \left\{ \sum_{i=1}^{\infty} \mu[B_i] \ : \ B_i \in \mathbb{A},\ A \subset \bigcup_{i=1}^{\infty} B_i \right\}. \tag{2}$$

5. Let us define a *lattice animal of size* n to be a set of vertices $C \subset \mathbb{Z}^d$ such that $0 \in C$, C is connected, and $|C| = n$. Let A_n be the set of lattice animals of size n. In this exercise we show that $|A_n| \leq 4^{dn}$.
   (a) Consider $\mathbb{P}_p$, our percolation measure on $\mathbb{Z}^d$. Let $C_0 \subset \mathbb{Z}^d$ be the cluster of the random percolation configuration under $\mathbb{P}_p$ which contains the origin. For a fixed lattice animal C of size n, show that

$$\mathbb{P}_p[C_0 = C] \geq p^{2dn}(1-p)^{2dn}. \tag{3}$$

(b) Prove that $|A_n| \leq 4^{dn}$.

6. (a) In the lectures we defined bond percolation. Give a definition of site percolation on $\mathbb{Z}^d$, where one uses vertices rather than edges.

   (b) Show that bond percolation on the two-dimensional lattice $\mathbb{Z}^2$ is equivalent to site percolation on a modified lattice.

   (c) Show that on $\mathbb{Z}^2$, $p_c(bond) \leq p_c(site)$.