Percolation exercises 2

1. (a) Let C_n be the number of self-avoiding walks of length n in $\mathbb{Z}^d$ starting at the origin. Show that

$$c_{n+m} \leq c_n \cdot c_m \tag{1}$$

and so one has $\log c_{n+m} \leq \log c_n + \log c_m$.

(b) Fix $k \in \mathbb{N}$. Using the decomposition $n = mk + r$, where $m, r \in \mathbb{N}$ with $1 \leq r \leq k$, and part (a), show that

$$\limsup_{n\to\infty} \frac{\log c_n}{n} \leq \frac{\log c_k}{k}. \tag{2}$$

(c) Deduce that $\lim_{n\to\infty} c_n^{1/n}$ exists and lies in $(0, \infty)$. This quantity is known as the *connective constant* of the graph $\mathbb{Z}^d$.

2. (a) Show that for $p > p_c$, $\theta(p) > 0$.

(b) In the lectures we defined a coupling $\mathbb{P}^*$ between the measures $\mathbb{P}_p$ and $\mathbb{P}_{p'}$, for $p < p'$. Let ω be the percolation process with law $\mathbb{P}_p$ and ω' the process with law $\mathbb{P}_{p'}$ defined in the coupling. Use the coupling to show that:

$$\mathbb{P}^*[\{0 \leftrightarrow \infty \text{ in } \omega'\} \cap \{0 \not\leftrightarrow \infty \text{ in } \omega\}] > 0. \tag{3}$$

(c) Deduce that $\theta(p)$ is strictly increasing on $(p_c, 1]$.

3. In this exercise we show that $\theta(p)$ is right continuous, that is, for all $p \in [0, 1]$,

$$\lim_{\varepsilon\to 0} \theta(p + \varepsilon) = \theta(p), \tag{4}$$

where here ε is always positive.

(a) Define $\theta_n(p) := \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$. Show that $\theta_n(p)$ is a continuous function of p, increasing in p and decreasing in n, and that $\lim_{n\to\infty} \theta_n(p) = \theta(p)$.

(b) Show that a decreasing limit of increasing functions from $[0, 1]$ to $\mathbb{R}$ is right continuous.

4. In this question, we fix Ω_1, Ω_2 to be sets, and $f : \Omega_1 \to \Omega_2$. If $\mathbb{A}_2 \subset 2^{\Omega_2}$, we define $f^{-1}(\mathbb{A}_2)$ to be the set of sets $f^{-1}(A)$ for all $A \in \mathbb{A}_2$.

(a) Show that:

- For $A, B \subset \Omega_2$, $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.
- For $A_i \subset \Omega_2$ $\forall i \in \mathbb{N}$, $f^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$;

(b) Let $\mathbb{A} \subset 2^{\Omega_2}$ a collection of subsets of Ω_2, and recall that $\sigma(\mathbb{A})$ is the smallest sigma-algebra containing $\mathbb{A}$. The aim of this part is to prove that $\sigma(\mathrm{f}^{-1}(\mathbb{A})) = f^{-1}(\sigma(\mathbb{A}))$.

(i) Show that since $\sigma(\mathbb{A})$ is a sigma algebra, so is $f^{-1}(\sigma(A))$. Further show that $f^{-1}(\mathbb{A}) \subset f^{-1}(\sigma(\mathbb{A}))$, and conclude that $\sigma(\mathrm{f}^{-1}(\mathbb{A})) \subset f^{-1}(\sigma(\mathbb{A}))$.

(ii) Let

$$\mathbb{D} = \{A \subset \Omega_2 \ : \ f^{-1}(A) \in \sigma(f^{-1}(\mathbb{A}))\}. \tag{5}$$

Show that $\mathbb{A} \subset \mathbb{D}$, and that $\mathbb{D}$ is a sigma algebra. Conclude that $\sigma(\mathrm{f}^{-1}(\mathbb{A})) \supset f^{-1}(\sigma(\mathbb{A}))$.

(c) Finally, let $\mathbb{F}_1 = \sigma(\mathbb{A}_1)$, $\mathbb{F}_2 = \sigma(\mathbb{A}_2)$ be sigma algebras on Ω_1, Ω_2, respectively. Show that if $f^{-1}(A) \in \mathbb{A}_1$ for all $A \in \mathbb{A}_2$, then f is measurable.