Percolation exercises 3

1. In this problem we give an alternative proof that $\theta(p) = \mathbb{P}_p[0 \leftrightarrow \infty]$ is continuous on $(p_c, 1]$. Drawing some pictures will probably help. Let $1 \leq k \leq n < \infty$. Let K be the number of disjoint clusters in Λ_n intersecting both Λ_k and $\partial\Lambda_n$. Let $U_{k,n}$ be the event that $K \leq 1$.
(a) Show that $\mathbb{P}[U_{k,n}]$ is nondecreasing in n and that since the infinite cluster is unique, $\mathbb{P}_p[U_{k,n}] \to 1$ as $n \to \infty$.
(b) The above shows that for all $\varepsilon > 0$ and for each $p \in [0, 1]$ and $k \in \mathbb{N}$, there is some $n = n(p, k, \varepsilon)$ such that
$$\mathbb{P}_p[U_{k,n}] > 1 - \varepsilon. \tag{1}$$
We want to show this uniformly in p, ie there exists $n = n(k, \varepsilon)$ independent of p such that (1) holds; show this, using the sets
$$O_m := \{p \in [0, 1] \ : \ \mathbb{P}_p[U_{k,m}] > 1 - \varepsilon\}, \tag{2}$$
and the compactness of $[0, 1]$.
(c) Recall we defined $\theta_n(p) := \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$. Let $p_1 > p_c$. We aim to show that θ_n converges uniformly to θ on $[p_1, 1]$. Using the events $\{0 \leftrightarrow \partial\Lambda_n\}$, $\{\Lambda_k \leftrightarrow \infty\}$, and $U_{k,n}$, show that for $k \geq 1$ large enough and $n \geq k$ large enough, we have, for all $p \in [p_1, 1]$,
$$\theta(p) \geq \theta_n(p) - 2\varepsilon. \tag{3}$$
(d) Deduce the uniform convergence.

2. In this exercise we prove the monotone convergence theorem. See the lecture notes on MyCourses for the definition of the integral of a function f, which we denote by $\mu(f)$. For $(x_n)_{n\in\mathbb{N}}, x \in [0, \infty]$, we say $x_n \nearrow x$ if $x_n \leq x_{n+1}$ and $x_n \to x$. Similarly, let Ω be a set, and let $f_n, f : \Omega \to [0, \infty]$. We say $f_n \nearrow f$ if $f_n(x) \nearrow f(x)$ for all $x \in \Omega$. The monotone convergence theorem says:

Theorem Let $(\Omega, \mathbb{F}, \mu)$ be a measure space. Let $f, (f_n)_{n\in\mathbb{N}}$ be non-negative, measurable functions on Ω, with $f_n \nearrow f$. Then $\mu(f_n) \nearrow \mu(f)$.
(a) Case 1: Prove the theorem for $f_n = \mathbb{1}_{A_n}$, $f = \mathbb{1}_A$, where A_n and A are events in $\mathbb{F}$.
(b) Case 2: Prove for f_n simple and $f = \mathbb{1}_A$. Use the sets
$$A_n = \{x \ : \ f_n(x) > 1 - \varepsilon\}, \tag{4}$$
where $\varepsilon > 0$.
(c) Case 3: Prove for f_n and f simple. Use the functions $a_k^{-1}\mathbb{1}_{A_k} f_n$, where $f =$

$\sum_{k=1}^{m} a_k \mathbb{A}_{\mathbb{k}}$ and linearity of the integral on simple functions.
(d) Case 4: Prove for f_n simple and $f \geq 0$ and measurable. For g simple, use the functions $\min\{f_n, g\}$, and the fact that on simple functions, the integral is monotone: $h_1 \leq h_2$ simple gives $\mu(h_1) \leq \mu(h_2)$.
(e) Case 5: Prove for f_n and f nonnegative and measurable. Use the functions

$$g_n := \min\{(2^{-n}\lfloor 2^n f_n \rfloor), n\}, \tag{5}$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$.

3. In this question, we study percolation on an infinite regular tree, and use a branching process (known as the Galton-Watson tree) to compute the critical point.

   The branching process describes the growth of some population. For $n \geq 1$, let Z_n be the number of individuals in the n^{th} generation. The population starts with one individual: $Z_0 = 1$. This individual has $k \geq 0$ children with some probability p_k, where $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. The vector $(p_k)_{k\geq 0}$ is called the *offspring distribution.*

   Each of the individuals in the first generation has, again, a random number of children given by $(p_k)_{k\geq 0}$, independently, and so on:

$$Z_{n+1} = \begin{cases} \sum_{i=1}^{Z_n} L_{n,i} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0, \end{cases} \tag{6}$$

   where $L_{n,i} \sim (p_k)_{k\geq 0}$ iid.

   We say the population goes *extinct* if $Z_n = 0$ for some $n \in \mathbb{N}$, and we say it *survives* if $Z_n > 0$ for all $n \in \mathbb{N}$. Survival/extinction depends on $(p_k)_{k\geq 0}$. To study this, we use generating functions. For $s \in [0, 1]$, define

$$f_n(s) := \mathbb{E}[s^{Z_n}]. \tag{7}$$

   (a) Compute f_0, f_1, $f_1'(1)$, and show f_1 is convex, that is, $f_1''(s) \geq 0$ for all $s \in [0, 1]$. (Here as $f : [0, 1] \to \mathbb{R}$, we think of $f'(1), f''(1)$, etc as the derivative from the left to make it well-defined. We also use the convention $0^0 = 1$.)
   (b) Show that

$$f_n(s) = \underbrace{f_1 \circ \cdots \circ f_1}_{n \text{ times}}(s). \tag{8}$$

   (c) Let $q \in (0, 1]$ be the smallest fixed point of f_1 (ie. the smallest element of $(0, 1]$ with $f_1(q) = q$. Show that

$$\lim_{n\to\infty} \mathbb{P}_p[Z_n = 0] = q. \tag{9}$$

   (d) Show that (assuming $p_1 < 1$) $\lim_{n\to\infty} \mathbb{P}_p[Z_n = 0] < 1$ (ie the population survives with positive probability) if and only if the expected number of offsprings of each individual is strictly greater than 1:

$$\sum_{k=0}^{\infty} k \cdot p_k > 1. \tag{10}$$

(e) Let $d \geq 1, d \in \mathbb{N}$. Consider an infinite tree T_d, such that every vertex has degree exactly $d+1$, apart from one chosen vertex, which we call the root, which has degree d. Consider percolation on T_d, and relate the cluster of the root to the branching process. Show that $p_c(T_d) = \frac{1}{d}$.