Percolation exercises 4

1. In this problem we'll give another proof of the Margulis-Russo formula: for all A increasing, $A \subset \{0,1\}^E$, E finite,

$$\frac{d}{dp}\mathbb{P}_p[A] = \sum_{e\in E} \mathbb{P}_p[e \text{ pivotal for } A]. \tag{1}$$

(a) Let $G = (V, E)$ be finite, and let $E = \{e_1, \ldots, e_n\}$. Let $\underline{p} = (p_1, \ldots, p_n) \in [0,1]^n$, and let $\omega^{\underline{p}} \in \{0,1\}^E$, satisfy $(\omega^{\underline{p}})_{e_i} = 1$ with probability p_i. Define such a random variable using the coupling measure $\mathbb{P}^*$, and show that for all $i = 1, \ldots, n$,

$$\frac{d}{dp_i}\mathbb{P}^*[\omega^{\underline{p}} \in A] = \mathbb{P}^*[e_i \text{ pivotal for } A \text{ in } \omega^{\underline{p}}]. \tag{2}$$

(b) Conclude the Margulis-Russo formula.

2. In the lectures in week 4, we will prove the FKG inequality: for all A, B increasing events, we have

$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B]. \tag{3}$$

Here we give an alternative proof for when A, B are dependent on finitely many edges. We work on a finite graph $G = (V, E)$. Let A, B be increasing events in $\{0,1\}^E$. For $\mathbb{P}_p[B] = 0$, the result is trivial, so we assume $\mathbb{P}[B] > 0$, and it suffices to prove

$$\mathbb{P}_p[A|B] \geq \mathbb{P}[A]. \tag{4}$$

(a) Let $e \in E$. We first prove the above for the case when $B = \{w_e = 1\}$. To do this, first show that

$$\frac{1}{p}\mathbb{P}_p[A, \omega_e = 1] \geq \frac{1}{1-p}\mathbb{P}_p[A, \omega_e = 0], \tag{5}$$

and then use this to show the case when $B = \{w_e = 1\}$.

(b) Let C be some event depending on edges in $E \setminus \{e\}$. Assume that we have

$$\mathbb{P}_p[A|C, \omega_e = 1] \geq \mathbb{P}_p[A|C]. \tag{6}$$

Show that $\mathbb{P}_p[\omega_e = 1|A, C] \geq \mathbb{P}_p[\omega_e = 1]$.

(c) Recall the coupling from the lectures. Let $E = \{e_1, \ldots, e_n\}$, and let U_{e_i} be iid uniform random variables on $[0,1]$ for $i = 1, \ldots, n$. Then $\omega \in \{0,1\}^E$ is defined by $\omega_{e_i} = 1$ iff $U_{e_i} \geq 1 - p$. Define another random variable $\eta \in \{0,1\}^E$ by

$$\eta_{e_i} = 1 \text{ iff } U_{e_i} \geq q_i, \tag{7}$$

where

$$
\begin{aligned}
q_1 &= \mathbb{P}^*[w_{e_1} = 1 \mid A], \\
q_i &= \mathbb{P}^*[w_{e_i} = 1 \mid A, \omega_{e_{[i-1]}} = \eta_{e_{[i-1]}}],
\end{aligned} \tag{8}
$$

where $\eta_{e_{[i-1]}} = (\eta_1, \ldots, \eta_{i-1})$. For clarity, we define

$$
\mathbb{P}^*[w_{e_i} = 1 \mid A, \omega_{e_{[i-1]}} = \eta_{e_{[i-1]}}] := \sum_{x \in \{0,1\}^{i-1}} \mathbb{P}^*[\eta_{e_{[i-1]}} = x]\mathbb{P}^*[w_{e_i} = 1 \mid A, \omega_{e_{[i-1]}} = x]. \tag{9}
$$

It is straightforward (if tedious) to check that $\mathbb{P}^*[\eta \in B] = \mathbb{P}^*[\omega \in B \mid \omega \in A]$ (you don't need to do this). Using part (b), show that $q_i \geq p$ for all $1 \leq i \leq n$, and conclude the FKG inequality.

3. In this question, we use another inequality, the BK-Reiner inequality, to give another proof of part 1 of the sharpness theorem. Let $A, B \in \{0,1\}^{E(\mathbb{Z}^d)}$ be events. We define an event $A \circ B$, which heuristically means that A and B both happen, but on disjoint sets of edges. For example, if $A = \{x \leftrightarrow y\}$ and $B = \{z \leftrightarrow w\}$, then $A \circ B$ is the event that there are two disjoint paths, one connecting x and y and one connecting z and w. Let us define $A \circ B$ precisely.

   For $\omega \in A$, we say $S \subset E(\mathbb{Z}^d)$ is a witness of A in ω if any other configuration $\omega' \in \{0,1\}^{E(\mathbb{Z}^d)}$ coinciding with ω on S is also in A. Then $A \circ B$ is the event that there exist witnesses $I = I(\omega)$ of A and $J = J(w)$ of B which are disjoint.

   The BK-Reiner inequality states: let A, B be cylinder events. Then

$$
\mathbb{P}_p[A \circ B] \leq \mathbb{P}_p[A]\mathbb{P}_p[B]. \tag{10}
$$

   We assume this holds for all cylinder events, as well as events $\{x \leftrightarrow y\}$ on $\mathbb{Z}^d$.

   In part 1 of the proof of sharpness (the part for $p < \tilde{p_c}$), we crucially used independence. Rewrite the proof such that it does not use independence, but uses the BK-Reiner inequality instead.