Percolation exercises 5

1. Let C_n be the event that there is an open crossing from one face of Λ_n to the opposite face. Use the square root trick and the event $U_{n,k}$ from exercise sheet 3 to show that when $\theta(p) > 0$,
$$\lim_{n\to\infty} \mathbb{P}_p[C_n] = 1. \tag{1}$$

2. Here we prove the BK-Reimer inequality for increasing events dependent on finitely many edges. Recall the definition of the events $\overline{A_i}$ and $\overline{B}$, and that in the lectures, we reduced the statement to proving that
$$\overline{\mathbb{P}_p}[\overline{A_{i-1}} \circ \overline{B}] \le \overline{\mathbb{P}_p}[\overline{A_i} \circ \overline{B}]. \tag{2}$$
Let $s : \overline{A_{i-1}} \circ \overline{B} \to \overline{A_i} \circ \overline{B}$ be defined as follows. If $\overline{\omega} \in \overline{A_{i-1}} \circ \overline{B}$ has witnesses I for $\overline{A_{i-1}}$ and J for $\overline{B}$ such that I does not contain e_i, then set $s(\overline{\omega}) = \overline{\omega}$. If all such witnesses I and J are such that $e_i \in I$, then obtain $s(\overline{\omega})$ by exchanging e_i and e_i', or more precisely:
$$s(\overline{\omega})(e) = \begin{cases} \overline{\omega}(e) & e \neq e_i, e_i' \\ \overline{\omega}(e_i') & e = e_i, \\ \overline{\omega}(e_i) & e = e_i'. \end{cases} \tag{3}$$
Prove that s is well-defined, is an injection, and is measure-preserving (that is, that $\overline{\mathbb{P}}_p[\overline{\omega}] = \overline{\mathbb{P}}_p[s(\overline{\omega})]$ for all $\overline{\omega} \in \overline{A_{i-1}} \circ \overline{B}$). Deduce that (2) holds.

3. Recall we defined $\theta_n(p) = \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$. In this question we show that $\theta_n(\frac{1}{2})$ decays polynomially in n. Let $R_n = [-n, n] \times [-n, n-1]$ as a subset of $\mathbb{Z}^2$, and let H_n be the event that there is an open path in ω from left to right in H_n. In the lectures we showed that $\mathbb{P}_{\frac{1}{2}}[H_n] = \frac{1}{2}$ for all $n \ge 1$.
(a) Show that
$$\theta_{2n}(\tfrac{1}{2}) \ge \frac{1}{4n}. \tag{4}$$
(b) Show that
$$\mathbb{P}_{\frac{1}{2}}[\{0 \leftrightarrow \partial\Lambda_n\} \circ \{0 \leftrightarrow \partial\Lambda_n\}] \ge \frac{1}{4n}, \tag{5}$$
and deduce a better lower bound for $\theta_n(\frac{1}{2})$ than in part (a).
(c) Let A_n be the event that there is an open circuit in ω^* in the annulus $\Lambda_{2n} \setminus \Lambda_n$ surrounding Λ_n. Assume we have that $\mathbb{P}_{\frac{1}{2}}[A_n] \ge c > 0$. Show that there is some $c' > 0$ such that
$$\theta_n(\tfrac{1}{2}) \le n^{-c'}. \tag{6}$$

4. In this question, we compute p_c for (bond) percolation on the triangular lattice $\mathbb{T}$ in

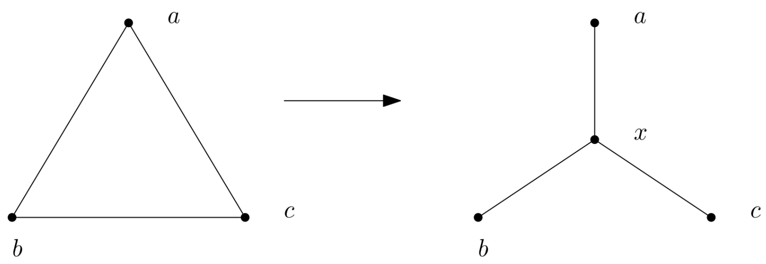

Figure 1

two dimensions.

(**The star-triangle transformation**) Let G be a planar graph and let a, b, c be vertices in G which form a triangle (that is, they are all connected by edges). Define G' to be the graph obtained from G by removing the edges ab, bc, and ac, adding a vertex x, and adding edges ax, bx, cx. (We say G' is obtained from G by the star-triangle transformation). See Figure 1.

(a) Let $\mathbb{P}_p$ be the Bernoulli percolation measure with parameter p on the edges of G, and let $\mathbb{P}'_{1-p}$ be the percolation on the edges of G' such that the parameter is $1-p$ on the edges ax, bx, and cx, and is p on all other edges. Show that if

$$p^3 + 1 = 3p, \tag{7}$$

one can define a coupling of $\mathbb{P}_p$ and $\mathbb{P}'_{1-p}$ on $\{0,1\}^{E(G)} \times \{0,1\}^{E(G')}$ such that probabilities of connections between vertices of G are the same in $\omega \sim \mathbb{P}_p$ and $\omega' \sim \mathbb{P}'_{1-p}$.
(b) Relate the triangular lattice $\mathbb{T}$ and the hexagonal lattice $\mathbb{H}$ by the star-triangle transformation. Show that if (7) holds, then $\mathbb{P}^{\mathbb{T}}_p[0 \leftrightarrow \infty] = \mathbb{P}^{\mathbb{H}}_{1-p}[0 \leftrightarrow \infty]$.
(c) Assume that our sharpness theorem holds for the lattices $\mathbb{T}$ and $\mathbb{H}$. Use duality to show that $p_c(\mathbb{T}) + p_c(\mathbb{H}) = 1$.
(d) Use parts (a), (b), and (c) to show that $p = p_c(\mathbb{T})$ is given by (7).