Percolation exercises 6

1. In this question we prove a crucial component of the conformal invariance proof. Drawing pictures may help.

   Let G be a $\mathbb{H}$-domain. Recall in the lectures we defined $\Omega_G = \{0,1\}^{F(G)}$, and Ω_G^{loop} as the set of loop configurations on G (subgraphs with every vertex having even degree). In the lectures in week 6 we will define the following. Write $E_{\text{mid}}(G)$ for the set of mid-points of edges of G. For $u, v, r, s \in E_{\text{mid}}(G)$, write

   - $\Omega_{G,uv}^{\text{loop}}$ for the set of configurations of disjoint loops on G together with a self-avoiding path $u \leftrightarrow v$, disjoint from the loops;
   - $\Omega_{G,uv,rs}^{\text{loop}}$ for the set of configurations of disjoint loops on G together with a two self-avoiding paths $u \leftrightarrow v$ and $r \leftrightarrow s$, disjoint from the loops and each other.

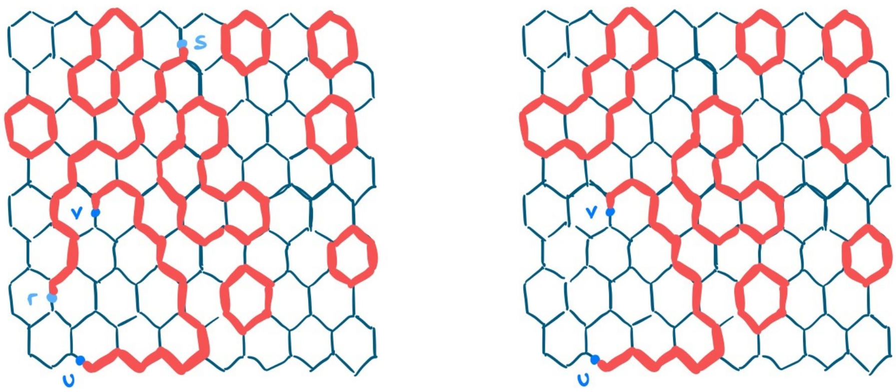

Figure 1: An example on the left of an element of $\Omega_{G,uv,rs}^{\text{loop}}$, and on the right of an element of $\Omega_{G,uv}^{\text{loop}}$

   See Figure 1. For the rest of the question, we let $a, b, c \in E_{\text{mid}}(G)$ lying on the boundary of G, and let $z \in E_{\text{mid}}(G)$.

   (a) Show that there is a bijection between $\Omega_{G,bc}^{\text{loop}}$ and Ω_G, such that the self-avoiding path γ joining b and c becomes the interface Γ in $\omega \in \Omega_G$ between open clusters touching the boundary arc (b, c) and closed clusters touching (c, b).

   (b) Let γ_0 be a fixed self-avoiding path $b \leftrightarrow c$. Let

   - $\Omega_{G,bc}^{\text{loop}}[\gamma_0]$ for the set of configurations $\eta \in \Omega_{G,uv}^{\text{loop}}$ whose self-avoiding path joining b and c is γ_0
   - and similar for $\Omega_{G,az,bc}^{\text{loop}}[\gamma_0]$.

Show that

$$\left|\Omega^{\text{loop}}_{G,az,bc}[\gamma_0]\right| = \begin{cases} 0 & \gamma_0 \text{ disconnects } a \text{ from } z \\ \left|\Omega^{\text{loop}}_{G,bc}[\gamma_0]\right| & \text{otherwise.} \end{cases} \tag{1}$$

(c) We now define

$$F_a(z) := \frac{\left|\Omega^{\text{loop}}_{G,az,bc}\right|}{|\Omega_G|}. \tag{2}$$

Use (a) and (b) to show that

$$F_a(z) = \mathbb{P}_{\frac{1}{2}}[\Gamma \text{ does not disconnect } a \text{ from } z], \tag{3}$$

where $\mathbb{P}_{\frac{1}{2}}$ is our percolation on the faces of G.

(d) Let $z, z' \in E_{\text{mid}}(G)$. Show that

$$|F_a(z) - F_a(z')| \leq \mathbb{P}_{\frac{1}{2}}[\Gamma \text{ disconnects exactly one of } z, z' \text{ from } a]. \tag{4}$$

(e) Let $\delta > 0$ and let $U \subset \mathbb{C}$ be a Jordan domain. All of the above works when $G = U_\delta := \delta\mathbb{T} \cap U$. For $q \in (0, 1]$, let Λ_q be the square of side-length q centred at 0. We focus on the case when $U = \Lambda_1$. From our work on RSW, one can show that there exists $\alpha \in (0, 1]$ and $c > 0$ such that for all $\delta > 0$ and $q \in (0, 1)$,

$$\mathbb{P}_{\frac{1}{2}}[\Lambda_q \overset{U_\delta}{\longleftrightarrow} \partial U_\delta] \leq c \cdot q^\alpha. \tag{5}$$

Show that in the case z, z' lie on opposite faces of $\partial\Lambda_q$ for some $q \in (0, 1)$, we have that for all $\delta > 0$,

$$|F_a(z) - F_a(z')| \leq c|z - z'|^\alpha. \tag{6}$$

2. In this question, we study a different model on $\mathbb{Z}^2$. A mirror at a vertex x is a line segment oriented at 45° to the lattice $\mathbb{Z}^2$ and centred at x. The orientation can be either north-west or north-east. Consider configurations ω where each vertex is assigned either a north-west mirror, or a north-east mirror, or no mirror.

Let $p \in [0, 1]$. The probability of $\mathbf{P}_p[\omega]$ is defined as follows. At each vertex x, independently place a north-west mirror with probability $p/2$, a north-east mirror with probability $p/2$ and no mirror with probability $1 - p$.

A configuration ω gives a collection of loops: each edge of $\mathbb{Z}^2$ is on exactly one loop, and at a vertex x, if there is a mirror then loops reflect off the mirror, whereas if there is no mirror, loops pass straight through the vertex and cross each other. See Figure 2, where one of the loops is highlighted in red.

We are interested in the event that 0 is connected by a loop to infinity, which we denote by $\{0 \leftrightarrow \infty\}$. It is conjectured that for all $p \in (0, 1]$,

$$\mathbf{P}_p[0 \leftrightarrow \infty] = 0. \tag{7}$$

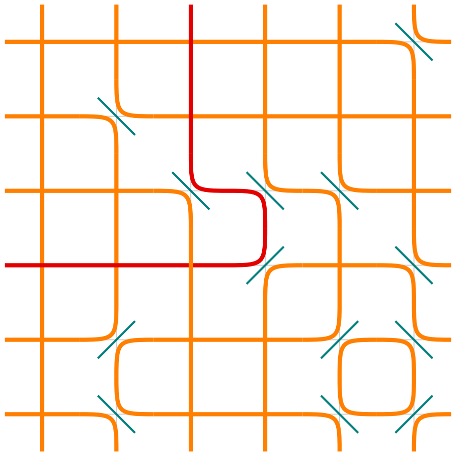

Figure 2: An example of a configuration ω, giving each vertex either a north-west mirror, or a north-east mirror, or no mirror.

(a) Interpreting mirrors as edges of a copy of $\mathbb{Z}^2$ or its dual, and using our knowledge about critical percolation, show that when $p = 1$, (7) holds.

(b) For $p < 1$, proving that (7) holds is an open problem. In the rest of this question, we will show that for all $p \in (0, 1]$

$$\mathbf{P}_p[0 \leftrightarrow \partial\Lambda_n] \geq \frac{1}{2n+1}. \tag{8}$$

Instead of on $\mathbb{Z}^2$, consider the model of mirrors above on an infinite cylinder $\mathbb{Z}_{2n+1} \times \mathbb{Z}$ of odd width $2n+1$ (this is a discrete circle of length $2n+1$ in the x direction, and $\mathbb{Z}$ in the y direction). Show that any configuration ω on this graph deterministically has an infinite path from $y = +\infty$ to $y = -\infty$.

(c) Fix some vertex in $\mathbb{Z}_{2n+1} \times \mathbb{Z}$ and call it 0. Show that on $\mathbb{Z}_{2n+1} \times \mathbb{Z}$,

$$\mathbf{P}_p[0 \leftrightarrow \infty] \geq \frac{1}{2n+1}, \tag{9}$$

and deduce that (8) holds on $\mathbb{Z}_{2n+1} \times \mathbb{Z}$.

(d) Use independence to show that this implies that (8) holds on $\mathbb{Z}^2$.

3. In this question we work with bond percolation on $\mathbb{Z}^2$. Drawing some pictures may help.

(a) Using our work on RSW, and the FKG inequality, show that there exists $c > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n] \leq c \cdot \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_{2n}]. \tag{10}$$

(b) Again using RSW and FKG, show that there exist $c, C > 0$ such that, for all $n \in \mathbb{N}$ and $x \in \partial\Lambda_n$,

$$c \cdot \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n]^2 \leq \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow x] \leq C \cdot \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n]^2, \tag{11}$$

where we use part (a) for the second inequality.

(c) Again using RSW and FKG, show that for all $n, N \in \mathbb{N}$ such that $2n \leq N$, there is a $c > 0$ such that

$$\mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_N] \leq \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_n]\mathbb{P}_{\frac{1}{2}}[\Lambda_n \leftrightarrow \partial\Lambda_N] \leq c \cdot \mathbb{P}_{\frac{1}{2}}[0 \leftrightarrow \partial\Lambda_N]. \tag{12}$$

(You won't need part (b) for this). This is known as *quasi-multiplicativity.* As a bonus (not required), can you replace $2n \leq N$ with $\alpha n \leq N$ for arbitrary $\alpha > 1$?