Lecture 2

Let K be a number field and S a finite set of places or F local field and

$\rho: G \to GL_n(k)$, $G = G_{K,S}$ or G_F.

Goal: a lift of ρ to a complete noetherian local ring A w/ residual field k is

$\rho_A: G \to GL_n(A)$ s.t
$$\begin{tikzcd} G \arrow[r, "\rho_A"] \arrow[dr, "\rho"'] & GL_n(A) \arrow[d, "red"] \\ & GL_n(k) \end{tikzcd}$$
$red \circ \rho_A = \rho$.

We want to understand all possible lifts of a given $\rho: G \to GL_n(k)$. In this lecture we make this precise.

A: noetherian local ring $\Rightarrow$ $\mathfrak{m}$ unique max ideal. $A/\mathfrak{m} = k$, residual field.

A is complete if $A = \varprojlim_{n \geq 1} A/\mathfrak{m}^n$. Notice $\{\mathfrak{m}^n\} \leftarrow$ basis of open ideals (f. index) w/ profinite topology

e.g. $\mathbb{Z}_p$

Why is this interesting? They play a crucial role in Taylor–Wiles.

$E/\mathbb{Q}$ with good reduction outside $S \supset \{p \mid p \mid \Delta_E\}$ we have $\rho_E: G_\mathbb{Q} \to GL_2(\mathbb{F}_p)$

p-torsion, $\rho_E: G_\mathbb{Q} \to GL_2(\mathbb{Z}/p^n\mathbb{Z})$ $\rightsquigarrow$ $\rho_E: G_\mathbb{Q} \to GL_2(\mathbb{Z}_p)$. $\forall \ell \notin S$,

$I_\ell \subseteq G_{\mathbb{Q}_\ell} \subseteq G_\mathbb{Q}$, $\rho_E(I_\ell) = Id$ i.e. ρ_E unramified at ℓ $\rightsquigarrow$ $\rho_E: G_{\mathbb{Q},S} \to GL_2(\mathbb{Z}_p)$

$$\begin{tikzcd} G_{\mathbb{Q},S} \arrow[r, "\rho_E"] \arrow[dr] & GL_2(\mathbb{Z}_p) \arrow[d, "red"] \\ & GL_2(\mathbb{F}_p) \end{tikzcd}$$
lift.

Later on: (Eichler–Shimura)

$f: \mathcal{H} \to \mathbb{C}$ holomorphic is weakly modular of weight k ~~and~~ for $\Gamma \subseteq SL_2(\mathbb{Z})$ f. index

if $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$.

$SL_2(\mathbb{Z})/\Gamma = \{\gamma_i \Gamma\}_{i=1}^{n}$ $\Rightarrow$ f is modular if at $\gamma_i(\infty)$ it "satisfies smoothness condition" $\Gamma = \Gamma_0(N) \Rightarrow f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}$ $z \in \mathcal{H}$.

cusp if $n=0$.

$M_K(N,R) \equiv$ modular forms of weight k for $\Gamma_0(N)\cap\Gamma_1(p)$ w/ coefs in R.

$$p\nmid N \Rightarrow T_p f(z) = \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + p^{k-1} f(pz)$$

$$p\mid N \Rightarrow U_p f(z) = \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right).$$ Hecke operators.

If f is T_p eigen-value $\Rightarrow$ $a_p(f) f = T_p f$.
$a_0(f) = 1$

<u>Eichler-Shimura</u>: f T_p-eigen $\forall p\nmid N$, weight 2, level N (for $\Gamma_0(N)$) $\Rightarrow$

$K_f = \mathbb{Q}(\{a_n(f)\}_{n\geq 1})$ number field. $\mathcal{O}_f$ ring of integers, $\mathfrak{p}\mid p$.

$\Rightarrow \exists\ \rho_{f,\mathfrak{p}}: G_{\mathbb{Q}} \to GL_2(\mathcal{O}_{f,\mathfrak{p}})$ s.t. $\forall \ell$ outside finitely many ($\ell\notin S = \{N, p\}$).

$\rho_{f,\mathfrak{p}}(I_\ell) = Id$ and $Tr(\rho_{f,\mathfrak{p}}(\Phi_\ell)) = a_\ell(f)$.
$\det(\rho_{f,\mathfrak{p}}(\Phi_\ell)) = \ell$.

• The deformation functor (Mazur)

$\Pi \equiv$ profinite (will be Galois), $\rho: \Pi \to GL_n(K)$. Can we describe all lifts to p-adically complete rings (suitable ones)?

<u>Condition Φ_p</u>: $\forall\ \Pi_0 \leq \Pi$ open of finite index $\exists$ finite $\lambda: \Pi_0 \to \mathbb{F}_p$ continuous homomorphs.

<u>e.g.</u>: Φ_p ok for $G_{\mathbb{Q}_\ell}$, $G_{\mathbb{Q},S}$.

Def: The pro-p completion of a profinite group Π is $\Pi^{(p)} := \varprojlim \Pi/N$
N closed normal of p-power index, continuous.

The p-Frattini quotient of Π is the maximal abelian quotient of Π with exponent p: is $\underset{Fr(\Pi)}{\underset{\shortparallel}{\Pi/M}}$ abelian, $\exists\, \phi: \Pi \to \Pi/M$ cont, $p\,Fr(\Pi) = e$

s.t is Π/N abelian, $\psi: \Pi \to \Pi/N$ cont, $p(\Pi/N) = e \Rightarrow \Pi/N \hookrightarrow \Pi/M$.

P.2.1. $Fr(\Pi)$ exists and it's image: $\Pi^{(p)} \to Fr(\Pi)$.

$\Pi^{ab}/p\Pi^{ab}$ abelian cont quotient of exp p + Zorn's lemma.

Lemma 2.1 Let Π_0 be profinite. TFSAE:

i) The pro-p completion of Π_0 is topologically f.g.

ii) The abelianisation of the pro-p completion is a $\mathbb{Z}_p$-module of finite rank.

iii) $Fr(\Pi_0)$ is finite.

iv) The set of cont. homomorphisms $\Pi_0 \to \mathbb{F}_p$ is finite.

Proof: A set of topological generators of the pro-p completion becomes a set of $\mathbb{Z}_p$-generators in the abelianisation and becomes a basis of p-Frattini: $(1) \Rightarrow (2) \Rightarrow (3)$.

Any $\Pi_0 \to \mathbb{F}_p$ cont factors through $Fr(\Pi_0)$ hence $(3) \Leftrightarrow (4)$.

Finally: pro-finite Burnside:

If the image in $Fr(\Pi_0)$ of $\{g_1, \dots, g_r\} \subseteq \Pi_0^{(p)}$ is basis of the quotient as vector space $/\mathbb{F}_p \Rightarrow \{g_1, \dots, g_r\}$ topologically generate $\Pi_0^{(p)}$ $(3) \Rightarrow (1)$ #

Note:

$\pi_0^{(p)} = \varprojlim_N \pi_0/N$ $\{f_1,\dots,f_n\}$ generate a dense subgroup in the pro-finite topology:

$\forall \alpha \in \pi_0^{(p)}$ $\alpha f_1^{a_1}\cdots f_n^{a_n} \in N$ $(a_1,\dots,a_n)$ coherent.

$\forall N$

$\Rightarrow$ in the abelianisation $\bar\alpha / \bar f_1^{a_1}\cdots \bar f_n^{-a_n} \in \bar N$ $\Rightarrow$ rk finite.

(a $\mathbb{Z}_p$-module)

$\pi_0^{ab}/M = Fr(\pi_0)$ ~~index~~ p-exponent $\Rightarrow$ $\bar\alpha M$ generated by $\bar f_1 M,\dots,\bar f_n M$ a_i red mod p

$\Rightarrow$ $\mathbb{F}_p$-vector space finite.

$$\begin{tikzcd} \varphi: \pi_0 \arrow[r] \arrow[d] & \mathbb{F}_p \\ Fr(\pi_0) \arrow[ur] & \end{tikzcd}$$

(3) $\Leftrightarrow$ (4).

Burnside: if the image of $\{g_1,\dots,g_r\}$ in $Fr(\pi_0^{(p)})$ basis of $\mathbb{F}_p$-v.s $\Rightarrow$ these top. gen. of $\pi_0^{(p)}$ #

Now: start w/ $\bar\rho: \pi \to GL_n(k)$, consider lift

$$\begin{tikzcd} \pi \arrow[r, "\rho"] \arrow[dr, "\bar\rho"'] & GL_n(R) \arrow[d, "\pi"] \\ & GL_n(\pi) \end{tikzcd}$$

$\mathcal{C}$ = category of complete noetherian local rings w/ res. field k.

morphism: $f\ R_1 \to R_2$ local homomorphisms of complete noeth. local rings inducing id on k.

$f(m_{R_1}) \subseteq m_{R_2}$

coeff. rings.

$\mathcal{C}^0$ = artinian local w/ res. field k (m in artinian local $\Rightarrow$ nilpotent hence noetherian

Artinian: $I_1 \supseteq I_2 \supseteq \dots$ stationary $\mathcal{C}^0 \subseteq \mathcal{C}$

m only max $m \supseteq m^2 \supseteq \dots$ stationary $\Rightarrow m^n = 0$.

P.2.3. objects of $\mathcal{C}$ are pro-objects of $\mathcal{C}^0$ R/m^n is Artinian.

P.2.4 topology of R/m^n is discrete

non-artinian element in $\mathcal{C}$: $W(k)$ = Witt vectors.

k finite $\Rightarrow$ unique unramified extn of $\mathbb{Z}_p$ whose residue field is k.

If $k = \mathbb{F}_p \Rightarrow W(k) = \mathbb{Z}_p$.

P.2.6 every coeff ring is an $W(k)$-algebra; i.e. $W(k) \to R$.

$\mathbb{Z}/p^n\mathbb{Z} \to R/m^n$

P.2.7 R coeff ring $\Rightarrow R \cong W(k)[[X_1 .. X_n]]/I$

$\Rightarrow W(k)/\pi W(k)^n \to R/m^n$.

Supp that ρ_f comes from a moduler form $\Rightarrow \bar{\rho}_f$ not only defined $/k$ but also

a particular lift to $\mathcal{O}$ dvr that maybe is not $W(k)$.

We decide if we want to restrict to coef rings which are $\mathcal{O}$-algebras.

$\Lambda \in \mathcal{C}$ i.e. complete noeth local w/ residual k.

$\mathcal{C}_\Lambda \equiv$ complete noeth. local Λ algebras w/ residual k

morph.: homs of coeff rings which are also Λ-algebra homomorphisms

$\mathcal{C}_\Lambda^0$ = artinian Λ-algebras $/k$.

$\mathcal{C} = \mathcal{C}_{W(k)}$.

Def: $\pi: R \to k \rightsquigarrow \pi: GL_n(R) \to GL_n(k)$ $\quad \Gamma_n(R) = \mathrm{Ker}\{GL_n(R) \xrightarrow{\pi} GL_n(k)\}$

$\rho_1, \rho_2 : \Pi \to GL_n(R)$ are strictly equivalent if $\exists M \in \Gamma_n(R)$ s.t. $\rho_1 = M\rho_2 M^{-1}$.

$\bar{\rho} : \Pi \to GL_n(k)$ residual rep. A deformation of $\bar{\rho}$ to R is a strict

equivalence class of lifts $\{M\rho M^{-1}, M \in \Gamma_n(R)\}$ $\bar{\rho} \to \bar{\rho}$.

$D = D_{\bar{\rho}} : \mathcal{C} \to \mathrm{Sets}$ $\quad$ deformation functor

$R \mapsto \{\text{defs of } \bar{\rho} \text{ to } R\}$ $\quad D_{\Lambda,\bar{\rho}} : \mathcal{C}_\Lambda \to \mathrm{Sets}$.

objects of $\mathcal{C}$ are pro-objects of $\mathcal{C}_0$: $R \in \mathcal{C} \Rightarrow R = \varprojlim R/\mathfrak{m}^k$

$R/\mathfrak{m}^k \in \mathcal{C}_0$.

$\Rightarrow \mathbb{F} : \mathcal{C} \to \text{Sets}$ $\quad$ $\mathbb{F}(R/\mathfrak{m}^k)$ form an inverse system.

$$\mathbb{F}(R/\mathfrak{m}^k) \to \mathbb{F}(R/\mathfrak{m}^{k'}) \qquad k' \leq k.$$

$\Rightarrow \mathbb{F}(R) \to \mathbb{F}(R/\mathfrak{m}^k)$ compatible $\Rightarrow \exists$ canonical $\mathbb{F}(R) \to \varprojlim_k \mathbb{F}(R/\mathfrak{m}^k)$

<u>Def</u>: A functor $\mathbb{F}$ on $\mathcal{C}$ is continuous if $\phi : \mathbb{F}(R) \xrightarrow{\sim} \varprojlim_k \mathbb{F}(R/\mathfrak{m}^k)$

<u>Lemma</u> 2.3 D and D_Λ are continuous functors.

$\Rightarrow D$ is completely determined by its values on $\mathcal{C}^0$. (Schlessinger)

Relative version: ρ_A lift to A.

look only at deformations of $\bar{\rho}$ which are defs of ρ_A. (to R s.t. $R \to A$)

i) Objects: coeff rings w/ A-augmentation ($R \to A$)

Mazur: A-augmented coeff rings (Λ a's) $\mathcal{C}(A)$

ii) strict equivalence: allow conj by matrices on $\operatorname{Ker}(GL_n(R) \to GL_n(A))$

$\rightsquigarrow \mathcal{C}^0_\Lambda(A)$.

<u>Rep functors</u> : $h_R : \mathcal{C} \to \text{Sets}$
$S \mapsto \operatorname{Hom}(R, S)$ ring homs + comp.

$\mathbb{F}$ rep if $\mathbb{F} \simeq h_R$ R, $D_{\bar{\rho}}(R) = \operatorname{Hom}(R_{\bar{\rho}}, R)$.

Lemma 2.3 $D_{\bar\rho}$, $D_{\bar\rho,\Lambda}$ are continuous.

Proof: we work for $D_{\bar\rho}$. Recall: $GL_n(R) = \varprojlim_k GL_n(R/m^k)$; $\Gamma_n(R) = \varprojlim_k \Gamma_n(R/m^k)$.

obs: $GL_n(R/m^{k+1}) \to GL_n(R/m^k) \to 1$

$\Gamma_n(R/m^{k+1}) \to GL_n(R/m^k) \to 1$.

we have to see that $D_{\bar\rho}(R) \cong \varprojlim D_{\bar\rho}(R/m^k)$ where $D_{\bar\rho}(R) \to \varprojlim_k D_{\bar\rho}(R/m^k)$

$\{\rho : \Pi \to GL_n(R)\} \mapsto \{\rho_k : \Pi \to GL_n(R/m^k)\}$

if def. were homomorphisms this would be obvious: but they are strict equivalence classes.

$\{M \rho M^{-1}\}$ $\rho : \Pi \to GL_n(R)$, $M \in \Gamma_n(R)$.

The canonical map $D_{\bar\rho}(R) \to \varprojlim_k D_{\bar\rho}(R/m^k)$

maps $\rho = \rho_R$ def. of $\bar\rho$ to R to $\{\rho_k\}$ where ρ_k is the def obtained by reducing ρ to m^k and the conj. is for $M \in \Gamma_n(R/m^k)$.

To show that the canonical map is surjective we need to show that any coherent $\{\rho_k\}$ comes from a deformation ρ to R. Enough to show that we can chose the homomorphisms representing ρ_k so as to have a coherent sequence of homomorphs.

For $k=1$, $\rho_1 = \bar\rho \Rightarrow$ there's no choice.

Assume chosen $r_1, \dots, r_k$ representing $\rho_1, \dots, \rho_k$. Let r' be any homomorph representing ρ_{k+1}.

$\Rightarrow \exists M_k \in \Gamma_n(R/m^k)$ s.t. $M_k^{-1}(r' \bmod m^k) M_k = r_k$.

Chose lift M_{k+1} of M_k to $\Gamma_n(R/m^{k+1})$, set $r_{k+1} = M_{k+1}^{-1} r' M_{k+1}$ extends the seq. to level $k+1$ $\rightsquigarrow \{r_k\}_{k \geq 1} : \Pi \to GL_n(R/m^k)$ coherent this gives:

~~$\rho : \Pi \to GL_n(R)$ $\rho \bmod m^k = \rho_k$~~ $\rho_k' = \rho'$ Canonical map is surj.

To prove injectivity: $\rho, \rho' : \Pi \to GL_n(R)$, $\rho_K \equiv \bar{\rho} \bmod m^K$
$\rho'_K \equiv \bar{\rho} \bmod m^K$

and ρ_K, ρ'_K are strictly equivalent are strictly equiv $\forall K \Rightarrow \rho, \rho'$ strictly eq.

i.e. $\forall K\ \exists M_K \in \Gamma_n(R/m^K)$ s.t. $\rho_K = M_K^{-1} \rho'_K M_K$. Choose M_K s.t. $M_{K+1} \equiv M_K\ \forall K$

so coherent, hence $M \in \Gamma(R)$, $\rho = M^{-1}\rho' M$ #.

<u>Universal def ring</u>

Assume $D_{\bar{\rho}}(R) = \mathrm{Hom}(R_{\bar{\rho}}, R)$ and transforms well under homs,

$$\forall R_1 \to R_2 \iff \mathrm{Hom}(R_{\bar{\rho}}, R_1) \to \mathrm{Hom}(R_{\bar{\rho}}, R_2)$$
$$f : R_{\bar{\rho}} \to R_1 \longmapsto R_{\bar{\rho}} \to R_1 \to R_2.$$

Consider $R = R_{\bar{\rho}}$, $\mathrm{Id} : R_{\bar{\rho}} \to R_{\bar{\rho}} \Rightarrow \exists\ \underset{\text{univ}}{\rho} : \Pi \to GL_n(R_{\bar{\rho}})$ universal.

$$\rho : \Pi \to GL_n(R) \Rightarrow \rho \rightsquigarrow \varphi : R_{\rho} \to R \qquad \rho_{univ} : \Pi \to GL_n(R_{\bar{\rho}})$$

$$\begin{array}{c} R_{\rho} \\ \uparrow \\ R_{\bar{\rho}} \end{array} \qquad \qquad \rho \searrow \downarrow \varphi \quad GL_n(R)$$

Given $\rho : \Pi \to GL_n(R)$ $\exists\ \varphi : R_{\bar{\rho}} \to R$ s.t. $\rho = \varphi \circ \rho_{univ}$.

<u>P.2.1.1.</u> Representable $\Rightarrow$ continuous.

<u>thm</u>: D representable by $R_{\bar{\rho}} \Rightarrow D_{\Lambda}$ representable by $R_{\Lambda} = R \hat{\otimes}_{W(k)} \Lambda$.

$R \mapsto \mathrm{Spec}(R)$. An A-valued point on $\mathrm{Spec}(R)$ is $R \xrightarrow{\varphi} A$ $\mathrm{Ker}(\varphi)$

$R \to A$ corresponds to deformations $\Rightarrow \mathrm{Spec}(R) \equiv$ def. space of $\bar{\rho}$.