Recall: $\mathcal{C} = \left\{ \begin{array}{l} \underline{\text{obj}}: \text{coeffient rings } / K \\ \underline{\text{Morph}}: \text{local homomorphism of local rings} \end{array} \right\}$ $\qquad \mathcal{C}^0 \subseteq \mathcal{C}$.

$\mathcal{C}^0 = \left\{ \begin{array}{l} \underline{\text{Obj}}: \text{artinian local rings, res. field } K \\ \underline{\text{Morph}}: \text{local homomorphisms of artinian local} \end{array} \right\}$

$\forall R \in \mathcal{C}, \quad R = \varprojlim_n \underbrace{R/\mathfrak{m}^n}_{\text{artinian} \in \mathcal{C}^0}$

- every coef ring $/K = \mathbb{F}_p$ is an $\mathbb{Z}_p$-algebra $\Rightarrow \mathbb{Z}_p \to R$

every coef ring $/ K = \mathbb{F}_q$ is an $\mathcal{O}_p$-algebra $\Rightarrow \mathcal{O}_p \to R \qquad \mathcal{O}_p / \mathbb{Z}_p$ finite.

$\mathcal{C}_\Lambda = \left\{ \begin{array}{l} \underline{\text{Obj}}: \text{coeff. rings, } \Lambda\text{-algebras } / K \\ \underline{\text{Morph}}: \text{local homs of local rings } \ \varphi(\lambda v) = \lambda \varphi(v),\ \lambda \in \Lambda \end{array} \right\}$

$\mathcal{C}^0_\Lambda \subseteq \mathcal{C}_\Lambda$ accordingly.

$\bar{\rho}: \Pi \to GL_n(K)$ deff of $\bar{\rho}$ to R is $\{ M \rho M^{-1} \mid M \in \Gamma_n(R) \}$

$$\begin{tikzcd} & \Pi \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}"'] & GL_n(R) \arrow[d, "\text{red mod } \mathfrak{m}"] \\ & & GL_n(K) \end{tikzcd}$$

$D = D_{\bar{\rho}}: \mathcal{C} \to \text{Sets}$

$\qquad R \longmapsto D_{\bar{\rho}}(R) = \{ \text{defs of } \bar{\rho} \text{ to } R \}$

$D = D_{\bar{\rho}, \Lambda}: \mathcal{C}_\Lambda \to \text{Sets}$ accordingly.

- Continuous functors: $\mathbb{F}: \mathcal{C} \to \text{Sets}, \ \forall R = \varprojlim R/\mathfrak{m}^k \in \mathcal{C} \quad R/\mathfrak{m}^{k_1} \to R/\mathfrak{m}^{k_2} \quad k_2 \leq k_1$

$\Rightarrow \mathbb{F}(R/\mathfrak{m}^{k_1}) \to \mathbb{F}(R/\mathfrak{m}^{k_2}) \Rightarrow \{ \mathbb{F}(R/\mathfrak{m}^k) \}_{k \geq 1}$ inverse system, with an

inverse limit $\varprojlim_k \mathbb{F}(R/\mathfrak{m}^k)$

Since $R \to R/\mathfrak{m}^k \Rightarrow \mathbb{F}(R) \to \mathbb{F}(R/\mathfrak{m}^k) \Rightarrow$ Canonical $\mathbb{F}(R) \to \varprojlim_k \mathbb{F}(R/\mathfrak{m}^k)$

Def: $\mathbb{F}$ is continuous if the canonical arrow is $\simeq$: $\boxed{\mathbb{F}(R) \xrightarrow{\sim} \varprojlim_k \mathbb{F}(R/\mathfrak{m}^k)}$

Lemma 2.3 $D_{\bar\rho}, D_{\bar\rho,\Lambda}$ are continuous functors.

Proof: (for $D_{\bar\rho}$)

Recall: $GL_n(R) = \varprojlim_k GL_n(R/m^k)$, $\Gamma_n(R) = \varprojlim_k \Gamma_n(R/m^k)$.

we have $GL_n(R/m^{k+1}) \to GL_n(R/m^k) \to 1$, $\Gamma_n(R/m^{k+1}) \to \Gamma_n(R/m^k) \to 1$.

$$D_{\bar\rho} : \mathcal{C} \to Sets$$
$$R \mapsto \{M\rho M^{-1} \mid M \in \Gamma_n(R)\}$$

if it were $D_{\bar\rho} : \mathcal{C} \to Sets$
$$R \mapsto \{\rho, \text{lift of } \bar\rho\}$$

$D_{\bar\rho}(R) = \{\rho, \text{lift of } \bar\rho \text{ to } R\}$

$D_{\bar\rho}(R/m^k) = \{\rho_k \text{ lift of } \bar\rho \text{ to } R/m^k\}$

$D_{\bar\rho}(R/m^{k+1}) \to D_{\bar\rho}(R/m^{k})$

$\Rightarrow D_{\bar\rho}(R) \xrightarrow{\sim} \varprojlim D_{\bar\rho}(R/m^k) \to 1.$

$\rho_1 : \Pi \to GL_n(R)$ $\quad \rho_{1,k} = \rho_{2,k} \ \forall k \Rightarrow \rho_1 = \rho_2$

$\rho_2 : \Pi \to GL_n(R)$

But there's the conjugacy issue: $D_{\bar\rho}(R) \to \varprojlim D_{\bar\rho}(R/m^k)$
$$\{M\rho M^{-1}\} \mapsto \{M_k \rho_k M_k^{-1}\}$$
$$\rho \mapsto \{\rho_k\}$$

This is surjective: we can chose the ρ_k's so as to have a coherent sequence: $\rho_1 = \bar\rho$ since $M_1 \in \Gamma_n(R/m) = \Gamma_n(k)$ $\bar\rho_1 = \bar\rho$ ~~$\Rightarrow M_1 = Id$~~.

Assume chosen $\{r_1, \dots, r_k\}$ representing the deformations $\{M_1\rho_1M_1^{-1}\}, \{M_2\rho_2M_2^{-1}\}, \dots$
$\{M_k\rho_kM_k^{-1}\}$.

Let r' represent $\{M_{k+1}\rho_{k+1}M_{k+1}^{-1}\} \Rightarrow \exists N_k \in \Gamma_n(R/m^k)$ s.t. $M_k r' M_k^{-1} \bmod m^k$
$= r_k$

lift M_{k+1} lifting M_k , $r_{k+1} = M_{k+1}\, r' M_{k+1}^{-1} \xrightarrow[\text{mod } k]{} M_k r_k M_k^{-1} \Rightarrow \{r_k\}$ gives

a deformation of $\bar\rho$ to R.

The canonical map is injective: $\rho, \rho' : \Pi \to GL_n(R)$ s.t. $\rho_k = \rho \bmod \mathfrak{m}^k$, $\rho'_k = \rho' \bmod \mathfrak{m}^k$

are strictly eq. $\forall k \Rightarrow \rho \simeq \rho'$.

$\exists M_k \in \Gamma_n(R/\mathfrak{m}^k)$ s.t. $\rho_k = M_k \rho'_k M_k^{-1}$, choose $M_k \equiv M_{k+1}\ \mathfrak{m}^k \Rightarrow \rho = M \rho' M^{-1}$ #

$\Rightarrow D_{\bar\rho}$ is determined by the values on $\mathcal{C}^0 \subseteq \mathcal{C}$. (schlessinger criteria).

· <u>Relative deformations</u>

Supp ρ_A lift of $\bar\rho$ to $A \in \mathcal{C}$. Can look at defs. of $\bar\rho$ which are deformations

of ρ_A i.e. $\rho : \Pi \to GL_n(R)$ $\quad R \xrightarrow{\varphi} A$.

$$\begin{tikzcd} \Pi \arrow[r, "\rho"] \arrow[dr, "\rho_A"] \arrow[ddr, "\bar\rho"'] & GL_n(R) \arrow[d, "\varphi"] \\ & GL_n(A) \arrow[d, "\bar\rho"] \\ & GL_n(k) \end{tikzcd}$$

- We work with category whose objects are coeff rings R with A-augmentation,

i.e. $R \to A$ (Λ algebra morph.): $\mathcal{C}(A)$, $\mathcal{C}_\Lambda(A)$.

- in the strict equivalence we conjugate by $\mathrm{Ker}[GL_n(R) \to GL_n(A)] \subseteq \Gamma_n(R)$

$\mathcal{C}(A) =$ A-augmented coeff rings.

· <u>Representable functors</u>

Want: $\exists R_{\bar\rho} \in \mathcal{C}$ s.t. $\forall R \in \mathcal{C}$, $D_{\bar\rho}(R) \cong \mathrm{Hom}(R_{\bar\rho}, R)$

$$\{\{M \rho M^{-1}\}\} \longrightarrow \{\varphi_\rho : R_{\bar\rho} \to R\}$$

in functorial way $\quad D_{\bar\rho}(R_1 \to R_2) \longmapsto \mathrm{Hom}(R_{\bar\rho}, R_1) \xrightarrow{\varphi_\rho} \mathrm{Hom}(R_{\bar\rho}, R_2)$

$$f \longmapsto \varphi_\rho \circ f.$$

Notice: Take $R_{\bar\rho} \in \mathcal{C}$ in case it exists

$$\begin{array}{ccc} D_{\bar\rho}(R_{\bar\rho}) & \xrightarrow{\sim} & \mathrm{Hom}(R_{\bar\rho}, R_{\bar\rho}) \\ \| & & \\ \{\{M\rho M^{-1}\}\} & & Id: R_{\bar\rho} \to R_{\bar\rho} \end{array}$$

Id corresponds to $\rho_{univ}: \Pi \to GL_n(R_{\bar\rho})$, $\forall \rho: \Pi \to GL_n(R)$,

~~Compose ρ_{univ}~~ Consider $D_{\bar\rho}(R) \cong \mathrm{Hom}(R, R_{\bar\rho})$

hence $\forall\ \rho: \Pi \to GL_n(R)$ $\exists\ \varphi: R \to R_{\bar\rho}$ s.t.

$$\begin{tikzcd} \Pi \arrow[r, "\rho_{univ}"] \arrow[dr] & GL_n(R_{\bar\rho}) \arrow[d, "\varphi"] \\ & GL_n(R) \end{tikzcd}$$

$\varphi \circ \rho_{univ} = \rho$

$$\begin{tikzcd} R_{\bar\rho} \arrow[r, "Id"'] \arrow[rr, bend right] & R_{\bar\rho} \arrow[r, "\varphi"'] & R \end{tikzcd}$$

$Id \circ \varphi$ corresponds to $\varphi \circ \rho_{univ}$.

$\|$

φ corresponds to ρ

$R_{\bar\rho}$: universal deformation ring.

P. 2.11: Every representable functor is continuous.

Thm 2.4: D representable by $R \Rightarrow D_\Lambda$ representable by $R_\Lambda := R \hat{\otimes}_{\mathbb{Z}_p} \Lambda$

The tensor product of 2 coeff ring need not be complete $\leadsto \hat{\otimes}$.

$R \xrightarrow{\xi} R_\Lambda$ induces ρ_Λ of $\bar\rho$ to R_Λ.

This is the ρ_{univ} for $\mathcal{C}_\Lambda$:

$$\begin{tikzcd} R \arrow[r, "\varphi_A"] \arrow[d, "\xi"] & A \\ R_\Lambda \arrow[ur, "\lambda_A"'] & \end{tikzcd}$$

$\Rightarrow \forall\ \rho: \Pi \to GL_n(A)$ $\supset \Lambda$-alg

$\exists\ \gamma$

$$\begin{tikzcd} \Pi \arrow[r] \arrow[dr] & GL_n(A) \\ & GL_n(R_\Lambda) \arrow[u, "\gamma"'] \end{tikzcd}$$

$\rho = \gamma \circ \rho_{univ}$

$\varphi_A \circ \rho_\Lambda = \rho_{\lambda_A}$

$= \cancel{\quad} \xi$

Supp $R, S \in \mathcal{C}$, $\mathrm{Hom}(R, S)$

$\Rightarrow \forall D$, $\mathrm{Hom}(D, A \times_C B) = \mathrm{Hom}(D,A) \times_{\mathrm{Hom}(D,C)} \mathrm{Hom}(D,B)$

$\Rightarrow$ Representable functors commute w/ fiber products: $\boxed{\mathbb{F}(A \times_C B) = \mathbb{F}(A) \times_{\mathbb{F}(C)} \mathbb{F}(B)}$

Mayer-Vietoris property (Mazur)

In general:

$$\begin{tikzcd} & A \times_C B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & C & \end{tikzcd} \Rightarrow \begin{tikzcd} & \mathbb{F}(A \times_C B) \arrow[dl, "\mathbb{F}(p)"'] \arrow[dr, "\mathbb{F}(q)"] & \\ \mathbb{F}(A) \arrow[dr, "\mathbb{F}(\alpha)"'] & & \mathbb{F}(B) \arrow[dl, "\mathbb{F}(\beta)"] \\ & \mathbb{F}(C) & \end{tikzcd} \Rightarrow \mathbb{F}(A \times_C B) \to \mathbb{F}(A) \times_{\mathbb{F}(C)} \mathbb{F}(B).$$

If fiber products exist $\Rightarrow$ Mayer-Vietoris is a necessary condition for being representable.

A, B, C comm rings $\Rightarrow$ $A \times_C B$ exist in the cat. of rings. Local is preserved

But fibered prod of noetherian is not necc. noetherian $\Rightarrow$ work in $\mathcal{C}^0$.

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P. 2.~~1~~19: $A, B, C \in \mathcal{C}^0 \Rightarrow A \times_C B \in \mathcal{C}^0$.

$\forall R \in \mathcal{C}$, $R = \varprojlim_{n} R/\mathfrak{m}^n$, $R/\mathfrak{m}^n \in \mathcal{C}^0$, If $\mathbb{F}$ is cont $\Rightarrow \mathbb{F}(R) = \varprojlim \mathbb{F}(R/\mathfrak{m}^n)$

May happen $\mathbb{F}$ not rep in $\mathcal{C}^0$ but $\exists R \in \mathcal{C}$ s.t. $\mathbb{F}(A) = \mathrm{Hom}(R, A)$

we say that $\mathbb{F}$ is $\boxed{\text{pro-representable}}$. Thee implies Mayer-Vietoris

~~Represen~~

$R_{\bar\rho}$ $\rightsquigarrow$ $\mathrm{Spec}(R_{\bar\rho}) = \{\mathfrak{p} \subseteq R_{\bar\rho} \text{ primes}\}$.

Def: A ring $X = \mathrm{Spec}(A)$.

$I \subseteq A$, $V(I) = \{\mathfrak{p} \in X \text{ s.t. } \mathfrak{p} \supseteq I\}$

~~$Y \subseteq A$ or $I \subseteq A$~~

$Y \subseteq X$, $J(Y) = \{f \in A \text{ s.t. } f \in \mathfrak{p}\ \forall \mathfrak{p} \in Y\}$.

| $A = \mathbb{C}[X,Y]$ $X = \mathrm{Spec}(A) = \{(x-x_0, y-y_0)\}$ + non max.

| $I \subseteq A$ $\cancel{X} = \{\mathfrak{p} \supseteq I = (f_1 .. f_r)\}$.
$V(I)$
$f_i(x-x_0, y-y_0) = 0$

The elements $X \setminus V(I)$ form the Zariski topology. It's not T_2 in general.

$\mathrm{Spec}(R_{\bar\rho})$ may have different connected components.

Def: A coeff ring, an A-valued point on $\mathrm{Spec}(R_{\bar\rho})$ is $\varphi : R_{\bar\rho} \to A$.

Rather, $\mathrm{Ker}(\varphi_A)$ ~~, which~~ is a prime ideal ~~$xy \in \mathrm{Ker}(\varphi_A) \Rightarrow \varphi_A(x)\varphi_A(y) = 0$~~
if A is a domain

However $\mathrm{Spec}(R_{\bar\rho})$ include all ring homomorphs $\varphi : R_{\bar\rho} \to A$ and we want only $\varphi : R_{\bar\rho} \to A \in \mathcal{C}$ $\rightsquigarrow$ must consider
$$\begin{array}{c}\mathrm{Specf}(R)\\ \downarrow\\ \mathrm{Specf}(\mathbb{Z}_p)\end{array}$$

$R = \mathbb{Z}_p[[X_1, X_2, X_3]]$ $\dim(R) = 4$, not 3, and we want dim 3 over $\mathbb{Z}_p$.

- Fiber products: $A, B, C \in \mathcal{C}$ $\alpha : A \to C$ $\beta : B \to C$

want
$$\begin{tikzcd} & A \times_C B \arrow[dl, dashed, "p"'] \arrow[dr, dashed, "q"] & \\ A \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & C & \end{tikzcd}$$
s.t. commutes and if $\exists$
$$\begin{tikzcd} & D \arrow[dl] \arrow[d] \arrow[dr] & \\ A \arrow[dr, "\alpha"'] & \beta & B \arrow[dl, "\beta"] \\ & C & \end{tikzcd} \to$$

$$\begin{tikzcd} & D \arrow[d, "\text{unique}"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & A \times_C B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & C & \end{tikzcd}$$

in sets
$A \times_C B = \{(a,b) \in A \times B \mid \alpha(a) = \beta(b)\}$

Def: $K[\varepsilon] = K[X]/(X^2)$ $\varepsilon = X \mod X^2$ dual numbers rings.

$\Lambda \to \Lambda/m_\Lambda = K \hookrightarrow K[\varepsilon]$ is Λ-algebra.

Thm (Grothendieck). Let $F : \mathcal{C}^0_\Lambda \to$ Sets covariant functor s.t. $F(K) = \{*\}$

$\Rightarrow$ F is pro-representable $\Leftrightarrow$ i) F satisfies Mayer-Vietoris ← difficult to check.

ii) $\#(K[\varepsilon])$ is finite.

(actually we'll see $F(K[\varepsilon])$ K-vector space)