

Recall: $\mathcal{C} = \left\{ \begin{array}{l} \text{Obj: coefficient rings } / K \\ \text{Morph: local homomorphism of local rings} \end{array} \right\}$

$\mathcal{C}^0 = \left\{ \begin{array}{l} \text{Obj: artinian local rings, res. field } K \\ \text{Morph: local homomorphism of artinian local} \end{array} \right\}$

$$\mathcal{C}^0 \subseteq \mathcal{C}$$

$$\forall R \in \mathcal{C}, \quad R = \varprojlim_{\mathfrak{m}^n} R/\mathfrak{m}^n \rightarrow \text{artinian} \in \mathcal{C}^0$$

every coeff ring $/K = \mathbb{F}_p$ is an \mathbb{Z}_p -algebra $\Rightarrow \mathbb{Z}_p \rightarrow R$

every coeff ring $/K = \mathbb{F}_q$ is an \mathcal{O}_p -algebra $\Rightarrow \mathcal{O}_p \rightarrow R$ $\mathcal{O}_p/\mathbb{Z}_p$ finite.

$\mathcal{C}_\Lambda = \left\{ \begin{array}{l} \text{Obj: coeff. rings, } \Lambda\text{-algebras } / K \\ \text{Morph: local maps of local rings } \varphi(\lambda v) = \lambda \varphi(v), \lambda \in \Lambda \end{array} \right\}$

$\mathcal{C}_\Lambda^0 \subseteq \mathcal{C}_\Lambda$ accordingly.

$\bar{P}: \Pi \rightarrow GL_n(K)$ def of \bar{P} to R is $\{M \bar{P} M^{-1} \mid M \in \Gamma_n(R)\}$

$$\begin{array}{ccc} P: \Pi \rightarrow GL_n(R) & & \\ \bar{P} \searrow & \downarrow \text{red mod } \mathfrak{m} & \\ & GL_n(K) & \end{array}$$

$D = D_{\bar{P}}: \mathcal{C} \rightarrow \text{Sets}$

$R \mapsto D_{\bar{P}}(R) = \{ \text{defts of } \bar{P} \text{ to } R \}$

$D = D_{\bar{P}, \Lambda}: \mathcal{C}_\Lambda \rightarrow \text{Sets}$ accordingly.

Continuous functor: $F: \mathcal{C} \rightarrow \text{Sets}$, $\forall R = \varprojlim_{\mathfrak{m}^k} R/\mathfrak{m}^k \in \mathcal{C}$ $R/\mathfrak{m}^{k_1} \rightarrow R/\mathfrak{m}^{k_2}$ $k_2 \leq k_1$

$\Rightarrow F(R/\mathfrak{m}^{k_1}) \rightarrow F(R/\mathfrak{m}^{k_2}) \Rightarrow \{F(R/\mathfrak{m}^k)\}_{k \geq 1}$ inverse system, with an

inverse limit $\varprojlim F(R/\mathfrak{m}^k)$

Since $R \rightarrow R/\mathfrak{m}^k \Rightarrow F(R) \rightarrow F(R/\mathfrak{m}^k) \Rightarrow \text{canonical } F(R) \rightarrow \varprojlim F(R/\mathfrak{m}^k)$

Def: F is continuous if the canonical arrow is \cong

$$\boxed{F(R) \xrightarrow{\cong} \varprojlim F(R/\mathfrak{m}^k)} \quad \square$$

Lemma 2.3 $D_{\bar{p}}, D_{\bar{p}, \lambda}$ are continuous functors.

Proof: (for $D_{\bar{p}}$)

Recall: $GL_n(R) = \varprojlim_K GL_n(R/\mathfrak{m}^k)$, $\Gamma_n(R) = \varprojlim_K \Gamma_n(R/\mathfrak{m}^k)$.

we have $GL_n(R/\mathfrak{m}^{k+1}) \rightarrow GL_n(R/\mathfrak{m}^k) \rightarrow 1$, $\Gamma_n(R/\mathfrak{m}^{k+1}) \rightarrow \Gamma_n(R/\mathfrak{m}^k) \rightarrow 1$.

$$D_{\bar{p}}: \mathcal{C} \rightarrow \text{Sets}$$

$$R \mapsto \{M \rho \bar{M}^{-1} \mid M \in \Gamma_n(R)\}$$

if it were $D_{\bar{p}}: \mathcal{C} \rightarrow \text{Sets}$
 $R \mapsto \{p, \text{lift of } \bar{p}\}$

$$D_{\bar{p}}(R) = \{p, \text{lift of } \bar{p} \text{ to } R\}$$

$$D_{\bar{p}}(R/\mathfrak{m}^k) = \{p_k, \text{lift of } \bar{p} \text{ to } R/\mathfrak{m}^k\}$$

$$D_{\bar{p}}(R/\mathfrak{m}^{k+1}) \rightarrow D_{\bar{p}}(R/\mathfrak{m}^k)$$

$$\Rightarrow D_{\bar{p}}(R) \cong \varprojlim_K D_{\bar{p}}(R/\mathfrak{m}^k) \rightarrow 1.$$

$$p_1: \pi \rightarrow GL_n(R)$$

$$p_{1,k} = p_{2,k} \quad \forall k \Rightarrow p_1 = p_2$$

$$p_2: \pi \rightarrow GL_n(R)$$

But there's the conjugacy issue: $D_{\bar{p}}(R) \rightarrow \varprojlim_K D_{\bar{p}}(R/\mathfrak{m}^k)$

$$\{M \rho \bar{M}^{-1}\} \rightarrow \{M_k \rho_k \bar{M}_k^{-1}\}$$

$$p \mapsto \{p_k\}$$

This is surjective: we can choose the p_k 's so as to have a coherent

sequence: $p_1 = \bar{p}$ since $M_1 \in \Gamma_n(R/\mathfrak{m}) = \Gamma_n(k)$ $\bar{p}_1 = \bar{p} \Rightarrow M_1 \in \text{Id}$.

Assume chosen $\{r_1, \dots, r_k\}$ representing the deformations $\{M_1 \rho_1 \bar{M}_1^{-1}\}, \{M_2 \rho_2 \bar{M}_2^{-1}\}, \dots$
 $\{M_k \rho_k \bar{M}_k^{-1}\}$.

let r' represent $\{M_{k+1} \rho_{k+1} \bar{M}_{k+1}^{-1}\} \Rightarrow \exists M_k \in \Gamma_n(R/\mathfrak{m}^k)$ s.t. $M_k r' \bar{M}_k^{-1} \text{ mod } \mathfrak{m}^k = r_k$

lift M_{k+1} lifting M_k , $r_{k+1} = M_{k+1} r_k^{-1} M_k^{-1} \rightarrow M_k r_k M_k^{-1} \equiv \{r_k\} \pmod{k}$ gives

a deformation of \bar{p} to R .

The canonical map is injective: $p, p': \Pi \rightarrow GL_n(R)$ s.t. $p_k = p \pmod{m_k}$
 $p'_k = p' \pmod{m_k}$
 are strictly eq. $\forall k \Rightarrow p \cong p'$

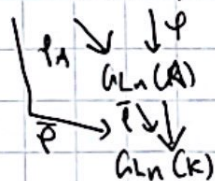
$\exists M_k \in \Gamma_n(R/m_k)$ s.t. $p_k = M_k p'_k M_k^{-1}$, choose $M_k \equiv M_{k+1} \pmod{m_k} \Rightarrow p = M p' M^{-1} \#$

$\Rightarrow D_{\bar{p}}$ is determined by the value on $\theta^0 \in \mathcal{G}$. (Schlessinger criteria).

Relative deformations

Supp p_A lift of \bar{p} to $A \in \mathcal{G}$. Can look at defs. of \bar{p} which are deformations

of p_A i.e. $p: \Pi \rightarrow GL_n(R)$ $R \xrightarrow{\varphi} A$.



• We work with category whose objects are coeff rings R with A -augmentation,

i.e. $R \rightarrow A$ (Λ algebra morph): $\mathcal{G}(A), \mathcal{G}_\Lambda(A)$.

• in the strict equivalence we conjugate by $\text{Ker}[GL_n(R) \rightarrow GL_n(A)] \in \Gamma_n(R)$

$\mathcal{G}(A) \cong A$ -augmented coeff rings.

Representable functors

Want: $\exists R_{\bar{p}} \in \mathcal{G}$ s.t. $\forall R \in \mathcal{G}, D_{\bar{p}}(R) \cong \text{Hom}(R_{\bar{p}}, R)$

$$\{ \{ M P M^{-1} \} \} \longrightarrow \{ \varphi_p: R_{\bar{p}} \rightarrow R \}$$

in functorial way

$$D_{\bar{p}}(R_1 \rightarrow R_2) \longmapsto \text{Hom}(R_{\bar{p}}, R_1) \xrightarrow{\varphi_p} \text{Hom}(R_{\bar{p}}, R_2) \longmapsto \varphi_p \# \quad \square$$

Notice: Take $R_{\bar{p}} \in \mathcal{C}$ in case it exists

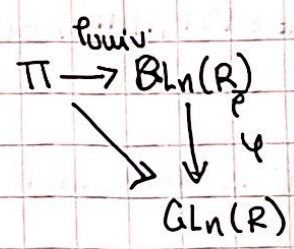
$$D_{\bar{p}}(R_{\bar{p}}) \xrightarrow{\sim} \text{Hom}(R_{\bar{p}}, R_{\bar{p}})$$

$$\int \left\{ \begin{array}{c} \text{Id} \\ \text{M} \\ \text{P} \\ \text{M}^{-1} \end{array} \right\} \quad \text{Id}: R_{\bar{p}} \rightarrow R_{\bar{p}}$$

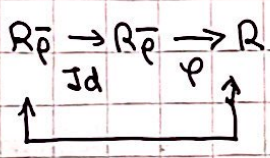
Id corresponds to $p_{\text{univ}}: \Pi \rightarrow \text{GL}_n(R_{\bar{p}})$, $\forall p: \Pi \rightarrow \text{GL}_n(R)$,

compose $\xrightarrow{p_{\text{univ}}}$ consider $D_{\bar{p}}(R) \cong \text{Hom}(R, R_{\bar{p}})$

hence $\forall p: \Pi \rightarrow \text{GL}_n(R) \exists \varphi: R \rightarrow R_{\bar{p}}$ s.t.



$$\varphi \circ p_{\text{univ}} = p$$



Id \circ φ corresponds to $\varphi \circ p_{\text{univ}}$
 \parallel
 φ corresponds to p

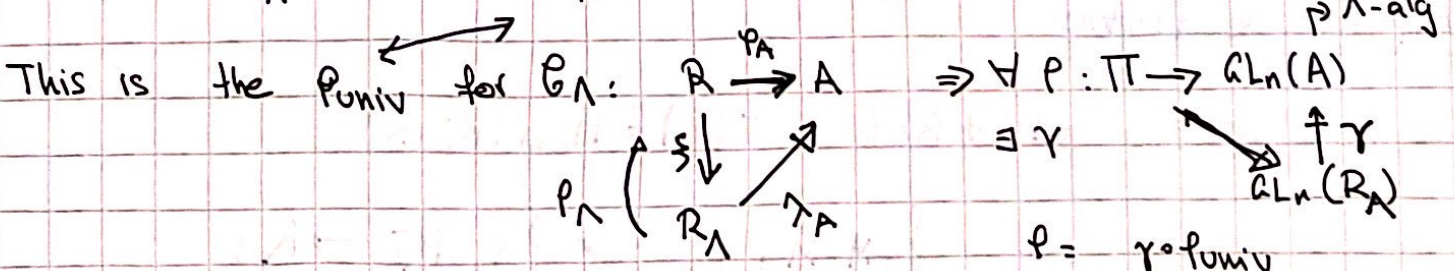
$R_{\bar{p}}$: universal deformation ring.

P.2.11: Every representable functor is continuous.

Thm 24: D representable by $R \Rightarrow D_{\Lambda}$ representable by $R_{\Lambda} := \widehat{R \otimes_{\mathbb{Z}_p} \Lambda}$

The tensor product of 2 coeff ring need not be complete $\text{no } \widehat{\otimes}$.

$R \xrightarrow{\xi} R_{\Lambda}$ induces p_{Λ} of \bar{p} to R_{Λ} .



$$\varphi_A \circ p_{\Lambda} = p_{\Lambda_A}$$

$$p = \gamma \circ p_{\text{univ}} = \gamma \circ p_{\Lambda}$$

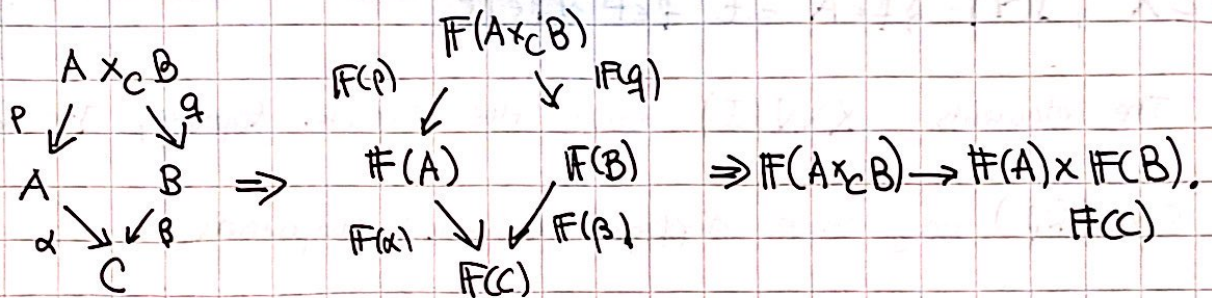
Supp $R, S \in \mathcal{C}$, $\text{Hom}(R, S)$

$$\Rightarrow \forall D, \text{Hom}(D, A \times_C B) = \text{Hom}(D, A) \times_{\text{Hom}(D, C)} \text{Hom}(D, B)$$

\Rightarrow Representable functors commute w/ fiber products: $\boxed{F(A \times_C B) = F(A) \times_{F(C)} F(B)}$

Mayer-Vietoris property
(Mazur)

In general:



If fiber products exist \Rightarrow Mayer-Vietoris is a necessary condition for being representable.

A, B, C comm rings $\Rightarrow A \times_C B$ exist in the cat. of rings. Local is preserved

But fibered prod of noetherian is not nec. noetherian \rightarrow work in \mathcal{C}^0 .

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P. 2.19: $A, B, C \in \mathcal{C}^0 \Rightarrow A \times_C B \in \mathcal{C}^0$

$\forall R \in \mathcal{C}, R = \varprojlim R/\mathfrak{m}^n, R/\mathfrak{m}^n \in \mathcal{C}^0$, If F is cont $\Rightarrow F(R) = \varprojlim F(R/\mathfrak{m}^n)$

May happen F not rep in \mathcal{C}^0 but $\exists R \in \mathcal{C}$ s.t. $F(A) = \text{Hom}(R, A)$

we say that F is pro-representable. This implies Mayer-Vietoris

Rephrase

$R_{\bar{p}} \mapsto \text{Spec}(R_{\bar{p}}) = \{ \mathfrak{p} \in R_{\bar{p}} \text{ primes} \}$

Def: A ring $X = \text{Spec}(A)$.

$I \subseteq A, V(I) = \{ \mathfrak{p} \in X \text{ s.t. } \mathfrak{p} \supseteq I \}$

$\forall \mathfrak{p} \in A \text{ or } I \in A$

$A = \mathbb{C}[X, Y] \quad X = \text{Spec}(A) = \{ (x-x_0, y-y_0) \}$ non max.
+
 $I \subseteq A \quad \mathfrak{p} = \{ \mathfrak{p} \supseteq I = (f_1, \dots, f_r) \}$
 $V(I) = \{ (x-x_0, y-y_0) \Rightarrow 0 \}$

$Y \subseteq X, I(Y) = \{ f \in A \text{ s.t. } f \in \mathfrak{p} \forall \mathfrak{p} \in Y \}$

The elements $X \setminus V(I)$ form the Zariski topology. It's not T_2 in general.

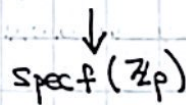
$\text{Spec}(R_{\bar{p}})$ may have different connected components.

Def: A coeff ring, an A -valued point on $\text{Spec}(R_{\bar{p}})$ is $\varphi: R_{\bar{p}} \rightarrow A$.

Rather, $\text{Ker}(\varphi_A)$, which is a prime ideal $\mathfrak{p} \in \text{Spec}(R_{\bar{p}}) \Rightarrow \varphi_A(x) = \varphi_A(y) = 0$
 if A is a domain

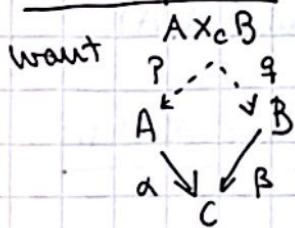
However $\text{Spec}(R_{\bar{p}})$ include all ring homomorphisms $\varphi: R_{\bar{p}} \rightarrow A$ and we

want only $\varphi: R_{\bar{p}} \rightarrow A \in \mathbb{C} \mapsto$ want consider $\text{Spec } \mathfrak{f}(R)$

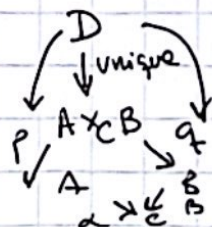
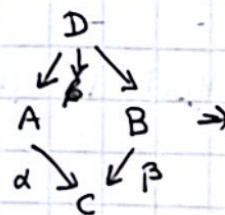


$R = \mathbb{Z}_p[[X_1, X_2, X_3]] \quad \dim(R) = 4$, not 3, and we want dim 3 over \mathbb{Z}_p .

• Fiber products: $A, B, C \in \mathcal{C} \quad \alpha: A \rightarrow C \quad \beta: B \rightarrow C$



s.t. commutes and if \exists



in sets

$$A \times_C B = \{ (a, b) \in A \times B \mid \alpha(a) = \beta(b) \}$$

Def: $K[\varepsilon] = K[X]/(X^2)$ $\varepsilon = X \bmod X^2$ dual numbers rings.

$\Lambda \rightarrow \Lambda/\mathfrak{m}_\Lambda = K \hookrightarrow K[\varepsilon]$ is Λ -algebra.

Thm (Grothendieck). Let $F: \mathcal{G}_\Lambda^0 \rightarrow \text{sets}$ covariant functor s.t. $F(K) = \{*\}$

$\Rightarrow F$ is pro-representable \Leftrightarrow i) F satisfies Mayer-Vietoris \leftarrow difficult to check.

ii) $\#(K[\varepsilon])$ is finite.

(actually we'll see $F(K[\varepsilon])$ K -vector space)