·The tangent space

$\Lambda \equiv$ coeff ring, $\mathcal{C}_\Lambda \equiv$ coeff Λ-algebras. $m_\Lambda \equiv$ max ideal of Λ.

$R \in \mathcal{C}_\Lambda$ m_R its max ideal

$$t_R^* = m_R / (m_R^2, m_\Lambda) \xrightarrow{} \text{Zariski cotangent}, \quad (m_R^2, m_\Lambda) = m_R^2 + i(m_\Lambda)R \equiv \text{"linealised functions}$$

which vanish at $(0,\dots,0)$".

<u>Obs</u>: t_R^* is a $\underbrace{\Lambda / m_\Lambda}_{\text{field } K}$ module via $i: \Lambda \to R$ $\Rightarrow$ it's a K-vector space
$i: \Lambda/m_\Lambda \to R/m_\Lambda R$.

<u>Def</u>: Zariski tangent space of R is

$$t_R := \mathrm{Hom}(t_R^*, K) \equiv \text{derivations}.$$

R noetherian $\Rightarrow m_R$ f.g $\Rightarrow t_R^*$ f.g. K-vector space $\Rightarrow t_R$ as well. (P.2.24)

<u>P.2.25</u> $f: B \to A \in \mathcal{C}_\Lambda \Rightarrow f_*: t_B^* \to t_A^*$ notice f surjective $\Leftrightarrow f_*$ injective

Let $F = \mathrm{Hom}(R, -)$ functor.

<u>Lemm</u>: There is a natural bijection $\mathrm{Hom}_K(t_R^*, K) \cong \mathrm{Hom}_\Lambda(R, K[\varepsilon])$:

<u>Proof</u>

A homomorphism of coeff Λ-algebras $R \to K[\varepsilon]$, since it induces Id on K is of the form $r \mapsto \bar{r} + \varphi(r)\varepsilon$, $r \equiv \bar{r} \mod m$, $\varphi(r) \in K$.

Since it's hom of Λ-algebras $\Rightarrow \varphi$ is determined by its image on m_R #

$\Rightarrow$ If $F = \mathrm{Hom}(R, -) \Rightarrow F(K[\varepsilon]) = t_R$ as sets.

Indeed, $\mathbb{F}(K[\varepsilon])$ is a K-vector space, (compatible with the bijection of lem 2.6).

$\alpha \in K$ gives: $\begin{aligned} K[\varepsilon] &\to K[\varepsilon] \\ a+b\varepsilon &\mapsto a+\alpha b\varepsilon \end{aligned}$ this is a ring hom and automorphism

$\Rightarrow$ by functoriality we have an automorphism $\mathbb{F}(K[\varepsilon]) \xrightarrow{\lambda_\alpha} \mathbb{F}(K[\varepsilon])$. This is scalar multiplication.

Addition: $\mathbb{F}$ representable $\Rightarrow$ Mayer-Vietoris:

$$\begin{tikzcd} & K[\varepsilon]\times_K K[\varepsilon] \arrow[dl] \arrow[dr] & \\ K[\varepsilon] \arrow[dr, "\text{proj.}"'] & & K[\varepsilon] \arrow[dl, "\text{proj.}"] \\ & K & \end{tikzcd}$$

$\exists!$ $R \to K$ Λ-algebra hom $\Rightarrow$ $\mathbb{F}(K)$ unipunctual $\Rightarrow$ $\mathbb{F}(K[\varepsilon])\times_{\mathbb{F}(K)}\mathbb{F}(K[\varepsilon])$

$$\simeq \mathbb{F}(K[\varepsilon])\times\mathbb{F}(K[\varepsilon])$$

$\underset{MV}{\Rightarrow}$ $\mathbb{F}(K[\varepsilon]\times_K K[\varepsilon]) \cong \mathbb{F}(K[\varepsilon])\times\mathbb{F}(K[\varepsilon])$

$\begin{aligned} A: K[\varepsilon]\times_K K[\varepsilon] &\to K[\varepsilon] \\ (x+y_1\varepsilon, x+y_2\varepsilon) &\mapsto x+(y_1+y_2)\varepsilon \end{aligned}$ $\Rightarrow$ $\mathbb{F}(K[\varepsilon])\times\mathbb{F}(K[\varepsilon]) \cong \mathbb{F}(K[\varepsilon]\times_K K[\varepsilon]) \xrightarrow{\mathbb{F}(A)} \mathbb{F}(K[\varepsilon])$

<u>Prop</u>: $\mathbb{F}: \mathcal{C}^0_\Lambda \to$ Sets covariant s.t. $\mathbb{F}(K)=\{*\}$

Suppose $\mathbb{F}(K[\varepsilon]\times_K K[\varepsilon]) \to \mathbb{F}(K[\varepsilon])\times\mathbb{F}(K[\varepsilon])$ is bijective

$\Rightarrow$ $\mathbb{F}(K[\varepsilon])$ has a natural K-vector space structure.

= tangent space hypothesis / K.

<u>Def</u>: $t_{\mathbb{F}} := \mathbb{F}(K[\varepsilon])$ tangent space of $\mathbb{F}$.

Goal: under suitable hypothesis, the $D_{\bar{\rho}}$ functor is pro-representable

κ = finite field of char p, Π profinite group with Φ_p-hypothesis.

$\bar{\rho} : \Pi \to GL_n(\kappa)$.

$\mathcal{C}_\Lambda = \begin{cases} \text{Obj: complete noeth } \Lambda\text{-algebras, res. field } \kappa \\ \text{Morph: local } \Lambda\text{-alg. homs s.t } 1_\kappa = Id. \end{cases}$

$D_{\bar{\rho}}$ = def of $\bar{\rho}$ to $\mathcal{C}$, $D_{\bar{\rho},\Lambda}$ = to $\mathcal{C}_\Lambda$

Schlessinger criteria

Regard $D_{\bar{\rho}}, D_{\bar{\rho},\Lambda} : \mathcal{C}^0 / \mathcal{C}^0_\Lambda \to$ Sets. Goal: these are prorep: $\exists R_{\bar{\rho}} \in \mathcal{C}$ s.t.

$D_{\bar{\rho}}(R) \cong Hom(R_{\bar{\rho}}, R)$ $\forall R \in \mathcal{C}^0$. (homs in $\mathcal{C}$).

Grothendieck: pro-rep is close to Mayer-Vietoris, but Schlessinger criteria are easier to deal with.

$F : \mathcal{C}^0_\Lambda \to$ Sets covariant.

Def: R, S coeff Λ-algebras $\phi : R \to S$ hom is small if it's surjective, $Ker(\phi)$ ppal and $m_R Ker(\phi) = \{0\}$.

P.3.1: $\phi : R \to S \to 0$ homomorph. in $\mathcal{C}^0_\Lambda \Rightarrow$
$$\begin{tikzcd} R \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & S \\ & T \arrow[ur, "i"'] & \end{tikzcd}$$
ψ, i small.

e.g. $\phi : \kappa[\varepsilon] \to \kappa \to 0$ $Ker(\phi) = \{a + b\varepsilon \mid a = 0\} = (\varepsilon)$

$m_{\kappa[\varepsilon]} = (\varepsilon)$, $m_{\kappa[\varepsilon]}(\varepsilon) = (\varepsilon^2) = (0)$.

Let $R_0, R_1, R_2 \in \mathcal{C}^0_\Lambda$ supp
$$\begin{tikzcd} R_1 \arrow[dr, "\phi_1"'] & & R_2 \arrow[dl, "\phi_2"] \\ & R_0 & \end{tikzcd}$$
in $\mathcal{C}^0_\Lambda$

$R_3 = R_1 \times_{R_0} R_2 = \{(r_1, r_2) \in R_1 \times R_2 \mid \phi_1(r_1) = \phi_2(r_2)\} \Rightarrow$ it's in $\mathcal{C}^0_\Lambda$.

$\rightsquigarrow$ $\circledast$ $\mathbb{F}(R_3) \longrightarrow \mathbb{F}(R_1) \times_{\mathbb{F}(R_0)} \mathbb{F}(R_2)$.

If $\mathbb{F}$ is rep $\Rightarrow$ $\circledast$ is bijective

if
$\circledast$ is a bijection if $R_1 = R_2 = K[\varepsilon]$, $R_0 = K$ $\Rightarrow$ $\mathbb{F}(K[\varepsilon])$ k-vector space.

Schlessinger conditions are weakened Grothendieck's.

H_1: If $R_2 \xrightarrow{\phi_0} R_0$ small $\Rightarrow$ $\circledast$ surjective

H_2: If $R_0 = K$ and $R_2 = K[\varepsilon]$ $\Rightarrow$ $\circledast$ bijective.

If H_2 ok $\Rightarrow$ for $R_1 = R_2 = K[\varepsilon]$ $\Rightarrow$ $t_F = \mathbb{F}(K[\varepsilon])$ k-vector space (tangent space condition) and call t_F the tangent space of $\mathbb{F}$.

H_3: $t_F = \mathbb{F}(K[\varepsilon])$ finite dim.

H_4 (M-V. variant): If $R_1 = R_2$, $\phi_i : R_i \to R_0$ same and small $\Rightarrow$ $\circledast$ bijective.

<u>Thm</u> (Schlessinger) Let $\mathbb{F} : \mathcal{C}^0_\Lambda \to$ Sets, $\mathbb{F}(K) = \{*\}$. If $\mathbb{F}$ satisfies $H_1, \dots, H_4$ $\Rightarrow$ $\mathbb{F}$ is pro-rep i.e. $\exists R \in \mathcal{C}_\Lambda$ s.t. $\mathbb{F}(A) = \mathrm{Hom}(R, A)$ $\forall A \in \mathcal{C}^0_A$.

$d = \dim_K t_{\mathbb{F}}$. R is an inverse limit of quotients of $\Lambda[[X_1, \dots, X_d]]$.

We'll often apply Schlessinger to a subfunctor of a representable functor.

$\mathbb{F}_1 : \mathcal{C}_\Lambda^\circ \to sets$ is a subfunctor of $\mathbb{F}$ if $\forall R \in \mathcal{C}_\Lambda^\circ$, $\mathbb{F}_1(R) \subseteq \mathbb{F}(R)$, more precisely

$\exists\ \mathbb{F}_1 \to \mathbb{F}$ which induces $\mathbb{F}_1(R) \subseteq \mathbb{F}(R)\ \forall R$.

<u>Prop 3.2.</u> Let $\mathbb{F}_1$ be a subfunctor of $\mathbb{F}$ s.t. $\mathbb{F}_1(K) = \mathbb{F}(K) = \{*\}$. Suppose $\mathbb{F}$ is

pro-rep and $\overset{\Rightarrow}{H_1 - \dots - H_4}$. If $\mathbb{F}_1$ satisfies $H_1 \Rightarrow \mathbb{F}_1$ satisfies all and hence

is pro-rep.

<u>P.3.2</u> Prove it

<u>P.3.4.</u> Conditions as in Prop 3.2. $R \in \mathcal{C}_\Lambda$ representing $\mathbb{F} \Rightarrow$ the object

representing $\mathbb{F}_1$ is a quotient of R.

$\mathbb{F}(R) \simeq \mathrm{Hom}(R, R)$, $\mathbb{F}_1(R) \simeq \mathrm{Hom}(R_1, R) \subseteq \mathrm{Hom}(R, R)$.

$\mathbb{F}_1(R_1) \simeq \underset{\substack{\cup \\ id: R_1 \to R_1}}{\mathrm{Hom}(R_1, R_1)} \subseteq \mathrm{Hom}(R, R_1) \Rightarrow \exists!\ \varphi: R \to R_1$ s.t. $\varphi|_{R_1} = Id$

$\Rightarrow \varphi$ surjects $\quad R \to R_1 \to 1$ and $\boxed{R/\mathrm{Ker}(\varphi) \simeq R_1}$

- <u>Mazur's alternative approach</u>

in any category where fiber products exist, we say that
$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ A \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & C & \end{tikzcd}$$
is cartesian

if the map $D \to A \times_C B$ is $\simeq$.

Supp $\mathbb{F}_1$ subfunctor of $\mathbb{F}: \mathcal{C}_\Lambda^\circ \to Sets$ covariant and $\mathbb{F}_1(K) = \mathbb{F}(K)$.

Given a diagram in $\mathcal{C}^0_\Lambda$
$$\begin{tikzcd} A \arrow[dr, "\alpha"'] & B \arrow[d, "\beta"] \\ & C \end{tikzcd}$$
consider

$$\begin{tikzcd} \mathbb{F}_1(A\times_C B) \arrow[r] \arrow[d, hook] & \mathbb{F}_1(A)\times_{\mathbb{F}_1(C)}\mathbb{F}_1(B) \arrow[d, hook] \\ \mathbb{F}(A\times_C B) \arrow[r] & \mathbb{F}(A)\times_{\mathbb{F}(C)}\mathbb{F}(B) \end{tikzcd}$$

If every such diagram is cartesian, Mazur says $\mathbb{F}_1 \subseteq \mathbb{F}$ is relatively rep.

$$\mathbb{F}_1(A\times_C B) \simeq \mathbb{F}(A\times_C B)\times_{\mathbb{F}(A)\times\mathbb{F}(C)\mathbb{F}(B)}\left(\mathbb{F}_1(A)\times_{\mathbb{F}(C)}\mathbb{F}_1(B)\right).$$

P.3.5. If $\mathbb{F}_1\subseteq\mathbb{F}$ relatively rep $\Rightarrow \forall i$, $\mathbb{F}_1$ satisfies H_i if $\mathbb{F}$ does.
likewise the tangent condition.

- Universal deformations exist

Apply Schlessinger to $D_{\Lambda,\bar\rho}: \mathcal{C}_\Lambda \to$ sets. $D_{\bar\rho,\Lambda}(R) = \{$defs of $\bar\rho$ to $R\}$.

The H_1, H_2, H_3 always hold. The fourth depends on $\bar\rho$.

Def: $\bar\rho$ residual, $C(\bar\rho) := \mathrm{Hom}_\Pi(k^n,k^n) = \{P\in M_n(k)$ s.t. $P\bar\rho(g)=\bar\rho(g)P \ \forall g\in\Pi\}$

$\Rightarrow k^n$ is Π-module and $C(\bar\rho)$ the ring of Π-mod endomorphism.

If $A\in\mathcal{C}_\Lambda$, $\rho:\Pi\to GL_n(A)$ def to A, ρ makes A^n a Π-module

Def. $\bar\rho:\Pi\to GL_n(k)$, $\rho\in D_{\bar\rho}(A)$. $C_A(\rho) = \mathrm{Hom}_\Pi(A^n,A^n) = \{P\in M_n(A) \mid P\rho(g)=\rho(g)P$ $\forall g\in\Pi\}$

$C(\bar\rho) =: C_k(\bar\rho)$.
interested in $C(\bar\rho)=k$

Thm (Mazur, Ramakrishna) Π profinite $\Phi_p \Rightarrow$
$D_{\bar\rho,\Lambda}$ H_1,H_2,H_3. If $C(\bar\rho)=k \Rightarrow$ also H_4.