

The tangent space

$\Lambda \equiv$ coeff ring, $\mathcal{O}_\Lambda \equiv$ coeff Λ -algebras. $\mathfrak{m}_\Lambda \equiv$ max ideal of Λ .

$R \in \mathcal{O}_\Lambda$ \mathfrak{m}_R its max ideal \rightarrow Zariski cotangent

$$t_R^* = \mathfrak{m}_R / (\mathfrak{m}_R^2, \mathfrak{m}_\Lambda), \quad (\mathfrak{m}_R^2, \mathfrak{m}_\Lambda) = \mathfrak{m}_R^2 + i(\mathfrak{m}_R)R \equiv \text{linearised functions}$$

which vanish at $(0, \dots, 0)$.

obs: t_R^* is a $\Lambda / \mathfrak{m}_\Lambda$ module via $i: \Lambda \rightarrow R \Rightarrow$ it's a K -vector space
 $\underbrace{\Lambda / \mathfrak{m}_\Lambda}_{\text{field } K}$ $i: \Lambda / \mathfrak{m}_\Lambda \rightarrow R / \mathfrak{m}_R$

tangent space

Def: Zariski space of R is

$$t_R = \text{Hom}(t_R^*, K) \equiv \text{derivations.}$$

R noetherian $\Rightarrow \mathfrak{m}_R$ f.g. $\Rightarrow t_R^*$ f.g. K -vector space $\Rightarrow t_R$ as well. (P. 2.24)

P. 2.25 $f: B \rightarrow A \in \mathcal{O}_\Lambda \Rightarrow f_*: t_B^* \rightarrow t_A^*$ notice f_* surjective $\Leftrightarrow f_*$ injective

Let $F = \text{Hom}(R, -)$ functor.

Lemma: There is a natural bijection $\text{Hom}_K(t_R^*, K) \cong \text{Hom}_\Lambda(R, K[\epsilon])$:

Proof

A homomorphism of coeff Λ -algebras $R \rightarrow K[\epsilon]$, since it induces Id on K is

of the form $r \mapsto \bar{r} + \varphi(r)\epsilon$, $r \equiv \bar{r} \pmod{\mathfrak{m}}$, $\varphi(r) \in K$.

Since it's hom of Λ -algebras $\Rightarrow \varphi$ is determined by its image on $\mathfrak{m}_R \neq$

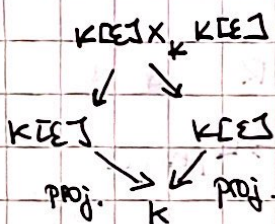
\Rightarrow if $F = \text{Hom}(R, -) \Rightarrow F(K[\epsilon]) = t_R$ as sets.

Indeed, $\mathbb{F}(K[E])$ is a K -vector space, (compatible with the bijection of lemma 2.6).

$d \in K$ gives: $K[E] \rightarrow K[E]$ this is a ring hom and automorphism
 $a+be \mapsto a+dbe$

\Rightarrow by functoriality we have an automorphism $\mathbb{F}(K[E]) \xrightarrow{d} \mathbb{F}(K[E])$. This is scalar multiplication.

Addition: \mathbb{F} representable \Rightarrow Mayer-Vietoris:



$\exists! R \rightarrow K$ Λ -algebra hom $\Rightarrow \mathbb{F}(K)$ unipunctal $\Rightarrow \mathbb{F}(K[E]) \times_{\mathbb{F}(K)} \mathbb{F}(K[E]) \simeq \mathbb{F}(K[E]) \times \mathbb{F}(K[E])$

$\Rightarrow \mathbb{F}(K[E] \times_K K[E]) \simeq \mathbb{F}(K[E]) \times \mathbb{F}(K[E])$
 MV

$K: K[E] \times_K K[E] \rightarrow K[E] \xrightarrow{\mathbb{F}(K)} \mathbb{F}(K[E]) \times_{\mathbb{F}(K)} \mathbb{F}(K[E]) \simeq \mathbb{F}(K[E] \times_K K[E]) \xrightarrow{\mathbb{F}(K)} \mathbb{F}(K[E])$
 $(x+y_1e, x+y_2e) \mapsto x+(y_1+y_2)e$

Prop. $\mathbb{F}: \mathcal{G}\Lambda^0 \rightarrow \text{Sets}$ covariant s.t. $\mathbb{F}(K) = \{*\}$

Suppose $\mathbb{F}(K[E] \times_K K[E]) \rightarrow \mathbb{F}(K[E]) \times \mathbb{F}(K[E])$ is bijective

$\Rightarrow \mathbb{F}(K[E])$ has a natural K -vector space structure.

\Rightarrow tangent space hypothesis (K).

Def. $t_{\mathbb{F}} := \mathbb{F}(K[E])$ tangent space of \mathbb{F} .

Goal: under suitable hypothesis, the $D\bar{\rho}$ functor is pro-representable

$\kappa \cong$ finite field of char p , Π profinite group with $\bar{\Phi}_p$ -hypothesis.

$$\rho: \Pi \rightarrow \mathrm{GL}_n(\kappa)$$

$\mathcal{C}_\Lambda = \{ \text{Obj: complete local } \Lambda\text{-algebras, res. field } \kappa \}$

Morph: local Λ -alg. homs s.t. $|_\kappa = \mathrm{Id}$.

$D\bar{\rho} \cong$ def of $\bar{\rho}$ to \mathcal{C} , $D\bar{\rho}_\Lambda \cong$ to \mathcal{C}_Λ

Schlessinger criteria

Regard $D\bar{\rho}, D\bar{\rho}_\Lambda: \mathcal{C}^0 / \mathcal{C}_\Lambda^0 \rightarrow \text{sets}$. Goal: these are prorep: $\exists R_\rho \in \mathcal{C}$ s.t.

$$D\bar{\rho}(R) \cong \mathrm{Hom}(R_\rho, R) \quad \forall R \in \mathcal{C}^0 \quad (\text{homs in } \mathcal{C}).$$

Grothendieck: pro rep is close to Mayer-Vietoris, but Schlessinger criteria are easier to deal with.

$\mathbb{F}: \mathcal{C}_\Lambda^0 \rightarrow \text{sets}$ covariant.

Def: R, S coeff Λ -algebras $\phi: R \rightarrow S$ hom is small if it's surjective,

$\mathrm{Ker}(\phi)$ ppal and $\mathbb{H}_R \mathrm{Ker}(\phi) = \{0\}$.

P.3.1: $\phi: R \rightarrow S \rightarrow 0$ homomorph. in $\mathcal{C}_\Lambda^0 \Rightarrow \begin{array}{ccc} R & \longrightarrow & S \\ \psi \searrow & & \nearrow i \\ & T & \end{array}$

ψ, i small.

e.g. $\phi: \kappa[\epsilon] \rightarrow \kappa \rightarrow 0$ $\mathrm{Ker}(\phi) = \{a + b\epsilon \mid a = 0\} = (\epsilon)$

$$\mathbb{H}_{\kappa[\epsilon]} = (\epsilon), \quad \mathbb{H}_{\kappa[\epsilon]}(\epsilon) = (\epsilon^2) = (0).$$

Let $R_0, R_1, R_2 \in \mathcal{P}_\Lambda^{\text{op}}$ s.t. $R_1 \xrightarrow{\phi_1} R_0 \xleftarrow{\phi_2} R_2$ in $\mathcal{P}_\Lambda^{\text{op}}$.

$R_0 = R_1 \times_{R_0} R_2 = \{ (r_1, r_2) \in R_1 \times R_2 \mid \phi_1(r_1) = \phi_2(r_2) \} \Rightarrow$ it's in $\mathcal{P}_\Lambda^{\text{op}}$.

$\text{Mod } \mathbb{F}(R_0) \rightarrow \text{Mod } \mathbb{F}(R_1) \times_{\text{Mod } \mathbb{F}(R_0)} \text{Mod } \mathbb{F}(R_2)$.

If \mathbb{F} is rep $\Rightarrow \otimes$ is bijective

\otimes is a bijection if $R_1 = R_2 = K[\mathbb{E}], R_0 = K \Rightarrow \mathbb{F}(K[\mathbb{E}])$ K -vector space.

Schlessinger conditions are weakened Grothendieck's.

H1: If $R_2 \xrightarrow{\phi_0} R_0$ small $\Rightarrow \otimes$ surjective

H2: If $R_0 = K$ and $R_2 = K[\mathbb{E}] \Rightarrow \otimes$ bijective.

If H2 ok \Rightarrow for $R_1 = R_2 = K[\mathbb{E}] \Rightarrow t_{\mathbb{F}} = \mathbb{F}(K[\mathbb{E}])$ K -vector space (tangent space condition) and call $t_{\mathbb{F}}$ the tangent space of \mathbb{F} .

H3: $t_{\mathbb{F}} = \mathbb{F}(K[\mathbb{E}])$ finite dim.

H4 (N-V. variant): If $R_1 = R_2$, $\phi_i: R_i \rightarrow R_0$ same and small

$\Rightarrow \otimes$ bijective.

Thm (Schlessinger) Let $\mathbb{F}: \mathcal{P}_\Lambda^{\text{op}} \rightarrow \text{sets}$, $\mathbb{F}(K) = \{ * \}$. If \mathbb{F} satisfies

H1, ..., H4 $\Rightarrow \mathbb{F}$ is pro-rep i.e. $\exists R \in \mathcal{P}_\Lambda^{\text{op}}$ s.t. $\mathbb{F}(A) = \text{Hom}(R, A) \forall A \in \mathcal{P}_\Lambda^{\text{op}}$.

$d = \dim_K t_{\mathbb{F}}$. R is an inverse limit of quotients of $\Lambda[[X_1, \dots, X_d]]$.

We'll often apply Schlessinger to a subfunctor of a representable functor.

$\mathbb{F}_1: \mathcal{C}_\Lambda^0 \rightarrow \text{sets}$ is a subfunctor of \mathbb{F} if $\forall R \in \mathcal{C}_\Lambda^0, \mathbb{F}_1(R) \subseteq \mathbb{F}(R)$, more precisely

$\exists \mathbb{F}_1 \rightarrow \mathbb{F}$ which induces $\mathbb{F}_1(R) \subseteq \mathbb{F}(R) \forall R$.

Prop 3.2. Let \mathbb{F}_1 be a subfunctor of \mathbb{F} s.t. $\mathbb{F}_1(k) = \mathbb{F}(k) = \{x, y\}$. Suppose \mathbb{F} is pro-rep and $\Rightarrow H_1 \dots H_4$. If \mathbb{F}_1 satisfies $H_1 \Rightarrow \mathbb{F}_1$ satisfies all and hence is pro-rep.

P.3.2 Prove it

P.3.4. Conditions as in Prop 3.2. $R \in \mathcal{C}_\Lambda$ representing $\mathbb{F} \Rightarrow$ the object representing \mathbb{F}_1 is a quotient of R .

$$\mathbb{F}(R) \cong \text{Hom}(R, R), \quad \mathbb{F}_1(R) \cong \text{Hom}(R_1, R) \subseteq \text{Hom}(R, R).$$

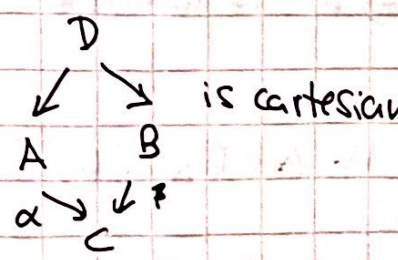
$$\mathbb{F}_1(R_1) \cong \text{Hom}(R_1, R_1) \subseteq \text{Hom}(R, R_1) \Rightarrow \exists! \psi: R \rightarrow R_1 \text{ s.t. } \psi|_{R_1} = \text{Id}$$

\downarrow
id: $R_1 \rightarrow R_1$

$$\Rightarrow \psi \text{ surjects} \quad R \rightarrow R_1 \rightarrow 1 \quad \text{and} \quad \overline{R/\text{Ker}(\psi)} \cong R_1$$

Mazur's alternative approach

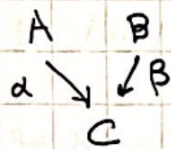
in any category where fiber products exist, we say that



if the map $D \rightarrow A \times_C B$ is \cong .

Supp \mathbb{F}_1 subfunctor of $\mathbb{F}: \mathcal{C}_\Lambda^0 \rightarrow \text{sets}$ covariant and $\mathbb{F}_1(k) = \mathbb{F}(k)$.

Given a diagram in \mathcal{B}_Λ^0



consider

$$\begin{array}{ccc} F_\Delta(A \times_C B) & \longrightarrow & F_\Delta(A) \times_{F_\Delta(C)} F_\Delta(B) \\ \downarrow & & \downarrow \\ F(A \times_C B) & \longrightarrow & F(A) \times_{F(C)} F(B) \end{array}$$

If every such diagram is cartesian, Mazur says $F_1 \subseteq F$ is relatively rep.

$$F_\Delta(A \times_C B) \simeq F(A \times_C B) \times_{F(A) \times_{F(C)} F(B)} (F_\Delta(A) \times_{F_\Delta(C)} F_\Delta(B))$$

P.3.5. If $F_1 \subseteq F$ relatively rep $\Rightarrow \forall i, F_1$ satisfies H_i if F does.

likewise the tangent condition.

• Universal deformations exist

Apply Schlessinger to $D_{\Lambda, \bar{p}}: \mathcal{B}_\Lambda \rightarrow \text{sets}$. $D_{\bar{p}, \Lambda}(R) = \{ \text{lifts of } \bar{p} \text{ to } R \}$

The H_1, H_2, H_3 always hold. The fourth depends on \bar{p} .

Def: \bar{p} residual, $C(\bar{p}) := \text{Hom}_\Pi(K^n, K^n) = \{ P \in M_n(K) \mid P \bar{p}(g) = \bar{p}(g) P \ \forall g \in \Gamma \}$

$\Rightarrow K^n$ is Π -module and $C(\bar{p})$ the ring of Π -mod endomorphism.

If $A \in \mathcal{B}_\Lambda$, $\rho: \Pi \rightarrow \text{GL}_n(A)$ def to A , ρ makes A^n a Π -module

Def: $\bar{\rho}: \Pi \rightarrow \text{GL}_n(K)$, $\rho \in D_{\bar{\rho}, \Lambda}(A)$. $C_A(\rho) = \text{Hom}_\Pi(A^n, A^n) = \{ P \in M_n(A) \mid P \rho(g) = \rho(g) P \ \forall g \in \Pi \}$

$C(\bar{\rho}) = C_K(\bar{\rho})$.

interested in $C(\bar{\rho}) = K$

Thm (Mazur, Iwasawa, Kurihara) Π profinite $\bar{\rho}_p \Rightarrow D_{\bar{\rho}_p, \Lambda} H_1, H_2, H_3$. If $C(\bar{\rho}) = K \Rightarrow$ also H_4 .