# Lecture 4 : Properties of the universal deformation

$\Pi$ profinite group w/ property $\Phi_p$. Typically $\Pi = G_{\mathbb{Q}_S}, G_{K S}$. $\bar{\rho}: \Pi \to GL_n(k)$ s.t. $C(\bar{\rho}) = k$.

- Functorial properties

det: $GL_n \to GL_1$ homomorphism of affine group schemes $/\mathbb{Z}$. It sends defs of $\bar{\rho}$ to defs of $\det(\bar{\rho}) \Rightarrow \exists \alpha: \Lambda[[\Gamma]] \to R_{\bar{\rho}}$ as $\Lambda$-algebras (coefficient), hence $R_{\bar{\rho}}$ is a $\Lambda[[\Gamma]]$-algebra.

e.g. Suppose $\bar{\rho} \sim \bar{\rho}'$ i.e. $\exists \bar{x} \in GL_n(k) \mid \bar{\rho} = \bar{x}^{-1}\bar{\rho}'\bar{x}$, $\exists x \in W(k) = \mathbb{Z}_p$ s.t. $x \to \bar{x}$.

Consider $R_{\bar{\rho}}$ and $R_{\bar{\rho}'}$ : $\rho_{univ}: \Pi \to GL_n(R_{\bar{\rho}}) \underset{x\text{-conjugation}}{\simeq} GL_n(R_{\bar{\rho}})$

$$\begin{tikzcd}
\rho_{univ}: \Pi \arrow[r] \arrow[dr] & GL_n(R_{\bar{\rho}}) \arrow[d] & GL_n(R_{\bar{\rho}'}) \arrow[dl, "\alpha"] \\
\text{def of } \bar{\rho}' \arrow[ur] & x^{-1}GL_n(R_{\bar{\rho}})x &
\end{tikzcd}$$

$x^{-1} \circ \rho_{univ} \cdot x$ def of $\bar{\rho}'$ to $R_{\bar{\rho}} \Rightarrow \exists \alpha: R_{\bar{\rho}'} \to R_{\bar{\rho}}$ s.t. $\alpha \circ \rho'_{univ} = x^{-1} \cdot \rho_{univ} \cdot x$

Now $x \rho'_{univ} x^{-1}$ def of $\bar{\rho}$ to $R_{\bar{\rho}'} \Rightarrow \exists \beta: R_{\bar{\rho}} \to R_{\bar{\rho}'}$ s.t $\beta \circ \rho_{univ} = x \rho'_{univ} x^{-1} \Rightarrow$ $\alpha \circ \rho'_{univ} = x^{-1} \rho_{univ} x = \alpha \circ (x^{-1} \cdot \beta \circ \rho_{univ} \cdot x) = x^{-1}(\alpha \circ \beta \circ \rho_{univ})x$

$\Rightarrow \rho_{univ} = \alpha \circ \beta \circ \rho_{univ} \Rightarrow \alpha \circ \beta = Id$ and $R_{\bar{\rho}'} \xrightarrow[\alpha]{\sim} R_{\bar{\rho}}$.

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P. 4.1 $\alpha$ does not depend on $x$, just on $\bar{\rho}, \bar{\rho}'$.

Def : (contragradient/transpose-inverse) rep : $\rho: \Pi \to GL_n(R)$,

define $\rho^*: \Pi \to GL_n(R)$

$g \mapsto \rho^*(g) = \rho(g^{-1})^t$. This is also called dual representation

Check : $\rho$ continuous $\Rightarrow \rho^*$ continuous

$C(\bar{\rho}) = k \Rightarrow C(\bar{\rho}^*) = k$.

**P.4.2.** $R_{\bar\rho^*} \equiv$ universal def. ring of $\bar\rho^*$ $\Rightarrow$ $R_{\bar\rho} \xrightarrow{\sim} R_{\bar\rho^*}$ canonically.

**P.4.3.** Supp $C(\bar\rho_1) = C(\bar\rho_2) = k$ and suppose $C(\bar\rho_1 \otimes \bar\rho_2) = k$. Given $\rho_i$ def of $\bar\rho_i$ to $R_i$ $\Rightarrow$ $\rho_1 \otimes \rho_2$ is a def of $\bar\rho_1 \otimes \bar\rho_2$ to $R_1 \hat\otimes R_2$ $\Rightarrow$

$\exists \alpha : R_{\bar\rho_1 \otimes \bar\rho_2} \to R_{\bar\rho_1} \hat\otimes R_{\bar\rho_2}$.

**P.4.4.** Notations as in P.4.3. Pick a lift $\rho_1$ of $\bar\rho_1$ to $GL_n(\Lambda)$ $\Rightarrow$

$\exists h_1 : R_{\bar\rho_1} \to \Lambda$ $\Rightarrow$ $\cancel{R_{\bar\rho_1} \otimes R_{\bar\rho_2} \to R_{\bar\rho_1}}$

$$\begin{tikzcd}
R_{\bar\rho_1 \otimes \bar\rho_2} \arrow[r, "\alpha"] \arrow[dr] & R_{\bar\rho_1} \hat\otimes R_{\bar\rho_2} \arrow[d, "h_1"] \\
& \Lambda \hat\otimes_\Lambda R_{\bar\rho_2} \cong R_{\bar\rho_2}
\end{tikzcd}$$

$h_1 \circ \alpha \equiv$ contraction with lift $\rho_1$.

If $\bar\rho_1$ is a character, $R_{\bar\rho_1 \otimes \bar\rho} \cong R_{\bar\rho}$ and $h_1$ is an isomorphism.

$$\begin{tikzcd}
R_{\bar\rho} \hat\otimes R_\chi \cong R_{\bar\rho} \hat\otimes_\Lambda \Lambda[[\Gamma]] \arrow[d, "h"] \arrow[d, shift left=8, "h_1"'] \\
\Lambda \hat\otimes_\Lambda R_{\bar\rho} \cong R_{\bar\rho}
\end{tikzcd}$$

$h_1 : \Lambda[[\Gamma]] \to \Lambda$

$\chi$ def of $\bar\chi$ to $\Lambda$

$\chi = h_1 \circ \chi_{univ}$.

$$\begin{tikzcd}
R_{\bar\rho} \hat\otimes \Lambda[[\Lambda]] \cong R_\rho \arrow[d, "\in Id \otimes h_1"] \\
R_{\bar\rho} \hat\otimes \Lambda \cong R_{\bar\rho}
\end{tikzcd}$$

is an isomorphism if

$Id \otimes h_1 \circ \rho_{univ, \bar\rho \otimes \bar\chi} = \rho_{univ, \bar\rho}$

given $\rho : \Pi \to GL_n(R)$ def of $\bar\rho$ to $R$. Consider $\underset{\alpha \circ \underline{\rho_{univ}}}{\rho} \otimes \underset{h_1 \circ \chi_{univ}}{\chi} =$

$(\alpha \otimes h_1)(\rho_{univ} \otimes \chi_{univ}) = (\alpha \otimes h_1) \circ \rho_{univ, \bar\rho \otimes \bar\chi} =$

$\alpha \circ (Id \otimes h_1) \circ \rho_{univ, \bar\rho \otimes \bar\chi}$ i.e. $\alpha : R_{\bar\rho} \to R$.

$\Rightarrow$ $(Id \circ h_1) \circ \rho_{univ, \bar\rho \otimes \bar\chi}$ satisfies the universal prop $\Rightarrow$ $(Id \circ h_1) \circ \rho_{univ, \bar\rho \otimes \bar\chi} = \rho_{univ}$.

• <u>Tangent space and cohomology groups</u>

$D := D_{\bar\rho,\Lambda}$. Recall $t_D = D(k[\varepsilon]) = \mathrm{Hom}_\Lambda(R_{\bar\rho}, k[\varepsilon]) \cong \mathrm{Hom}_k\left( m_{R_{\bar\rho}}/(m_R^2, m_\Lambda), k\right)$

Suppose $\rho_1 \in D_{\bar\rho,\Lambda}(k[\varepsilon]) \Rightarrow \rho_1(g) = (1 + b_g\varepsilon)a$, $\underset{\bar\rho(g)}{\overset{\shortparallel}{a}} \in GL_n(k)$, $b_g \in M_n(k)$.

Hence, $\rho_1$ determines and it's determined

by $\begin{aligned} b : \Pi &\longrightarrow M_n(k) \\ g &\longmapsto b_g \end{aligned}$ (i.e. $GL_n(k[\varepsilon]) \cong \underbrace{(I + \varepsilon M_n(k))}_{\cong (M_n(k),+)} \rtimes GL_n(k)$).

The fact that $\rho_1(gh) = \rho_1(g)\rho(h)$ it's equivalent to say that

$b : g \mapsto b_g$ is a <u>cocycle with values in $M_n(k) \circlearrowleft \Pi$</u> $= Ad(\bar\rho)$
conjugation

$g.b = \bar\rho(g).\, b\, \bar\rho(g)^{-1}$.

i.e. that $\boxed{b \in H^1(\Pi, \underset{M_n(k)}{\overset{\shortparallel}{Ad(\bar\rho)}})}$

$Adj(\bar\rho) : \Pi \longrightarrow GL_{n^2}(M_n(k))$
$g \longmapsto \bar\rho(g) : M \mapsto \bar\rho(g) M \bar\rho(g)^{-1}$

Yes: $\rho(g) = \bar\rho(g)(I + \varepsilon b_g)$, $\rho(h) = \bar\rho(h)(I + \varepsilon b_h)$, $\rho(gh) = \bar\rho(g)\bar\rho(h)(I + \varepsilon b_{gh})$

$\rho(gh) = \bar\rho(g)\bar\rho(h)(I + \varepsilon b_{gh}) = \rho(g)\rho(h) = \bar\rho(g)\bar\rho(h) + \varepsilon\bar\rho(g)\bar\rho(h) b_h + \varepsilon \bar\rho(g) b_g \bar\rho(h) + 0$
$= \bar\rho(g)\bar\rho(h) + \varepsilon \bar\rho(g)\bar\rho(h) b_{gh}$

$\overset{\Rightarrow}{\uparrow}$
$\cdot\, \bar\rho(h)^{-1}\rho(g)^{-1}$

$$b_{gh} = b_h + \bar\rho(h)^{-1} b_g \bar\rho(h) = b_h + h \circ b_g.$$

1-coboundary: $g \mapsto g.m - m$ for some $m$.

$\cancel{\tilde\rho(g) = \bar\rho(g)(I + \varepsilon b_g) = \gamma\rho(h)\gamma^{-1} = \gamma\bar\rho(h)(I + \varepsilon b_h)\gamma^{-1}}$

$\cancel{\Rightarrow \bar\rho(g) + \varepsilon\bar\rho(g) b_g = \bar\gamma\bar\rho(h)\bar\gamma^{-1} + \varepsilon\gamma\bar\rho(h) b_h \bar\gamma^{-1}}$

$\cancel{\Rightarrow \bar\rho(g) b_g = \gamma\bar\rho(h) b_h \bar\gamma^{-1}} \Rightarrow$

If $\rho_1 = \gamma \rho_2 \gamma^{-1}$, $\gamma \in \Gamma_n(K[\varepsilon]) \Rightarrow$ $b^1 : g \mapsto b^1_g$ differ by $g \mapsto gm - g$
$b^2 : g \mapsto b^2_g$ for some $m \in M_n(K)$.

$\rho_1(g) = \bar{\rho}(g)(I + \varepsilon b^1_g)$ $\rho_2(g) = \bar{\rho}(g)(I + \varepsilon b^2_g)$, $\gamma = I + \varepsilon M \Rightarrow \rho_1 \gamma = \gamma \rho_2 \Rightarrow$

$$(\bar{\rho}(g) + \varepsilon \bar{\rho}(g) b^1_g)(I + \varepsilon M) = (I + \varepsilon M)(\bar{\rho}(g) + \varepsilon \bar{\rho}(g) b^2_g) \Rightarrow$$

$$\cancel{\bar{\rho}(g)} + \varepsilon \bar{\rho}(g) M + \varepsilon \bar{\rho}(g) b^1_g = \cancel{\bar{\rho}(g)} + \varepsilon \bar{\rho}(g) b^2_g + \varepsilon M \bar{\rho}(g) \Rightarrow$$

$$M + b^1_g = b^2_g + \bar{\rho}(g)^{-1} M \bar{\rho}(g) \Rightarrow b^1_g = b^2_g + g.M - M$$

$\Rightarrow [b] \in H^1(\Pi, Ad(\bar{\rho}))$

About deformations, rather than lifts: $t_{D_{\bar{\rho},\Lambda}} \cong H^1(\Pi, Ad(\bar{\rho}))$ (P.47).

Cor 4.) Notations as before; let $d_1 := \dim_K H^1(\Pi, Ad(\bar{\rho})) = \dim t_{D_{\bar{\rho},\Lambda}}$.

$\Rightarrow R$ is a quotient of $\Lambda[[X_1, \dots, X_{d_1}]]$.

Proof: $t_{D_{\bar{\rho},\Lambda}} (= D_{\bar{\rho},\Lambda}(K[\varepsilon])) = Hom_\Lambda(R_{\bar{\rho}}, K[\varepsilon]) \cong Hom_K((m_R/(m_R^2, m_\Lambda), K)$

$\Rightarrow m_R/(m_R^2, m_\Lambda)$ has dim $d_1$ as subspace $\Rightarrow m_R$ is generated

by $d_1$ elements as $R$-module (Nakayama) $J(R) = m_R$.

P. 27 (see proof given in class) $\Rightarrow m = (y_1, \dots, y_{d_1}) \Rightarrow$

$$0 \to I \to \mathbb{Z}_p[[X_1, \dots, X_{d_1}]] \to R \to 0$$
$$x_i \mapsto y_i$$

$\Lambda$ torsion free as $\mathbb{Z}_p$-module $\Rightarrow$ this is exact $\Rightarrow 0 \to I \otimes \Lambda \to \Lambda[[X_1, \dots X_{d_1}]] \to R \otimes \Lambda \to 0$ ($= R$)

Prove it in general.

· Extensions of modules

Def : An extension of $\bar{\rho}$ by $\bar{\rho}$ is a vector space $E \circlearrowleft \Pi$ s.t.

$$0 \to V_{\bar{\rho}} \to E \to V_{\bar{\rho}} \to 0 \quad \text{as } K[[\Pi]]\text{-modules.}$$

$\mathrm{Ext}^1_{K[[\Pi]]}(V_{\bar{\rho}}, V_{\bar{\rho}})$ = extensions of $\bar{\rho}$ by $\bar{\rho}$.

this has a $K$-vector space structure non-triviall. (P.4.11).

Prop : There exists a bijection $D(K[\varepsilon]) \simeq \mathrm{Ext}^1_{K[[\Pi]]}(V_{\bar{\rho}}, V_{\bar{\rho}})$.
i.e. $\rho : \Pi \to GL_n(K[\varepsilon])$

Proof: For $\rho \in D(K[\varepsilon])$, $M := K[\varepsilon]^n$ with $\Pi$-action given by $\rho \Rightarrow$

$M$ $K$-dim $= 2n$. Consider $\varepsilon M \subseteq M$ and $M/\varepsilon M$ these have dim $n$

and $\simeq V_{\bar{\rho}}^2$ as $\Pi$-modules (check: P.4.8)

Hence $0 \to V_{\bar{\rho}} \simeq \varepsilon M \to E = M \to M/\varepsilon M \simeq V_{\bar{\rho}} \to 0$

Reciprocally, given $0 \to V_{\bar{\rho}} \xrightarrow{\alpha} E \xrightarrow{\beta} V_{\bar{\rho}} \to 0 \Rightarrow E$ is a $K[\varepsilon]$-module via #
$\varepsilon = \alpha \circ \beta$

Moreover: given $0 \to V_{\bar{\rho}} \xrightarrow{\alpha} E \xrightarrow{\beta} V_{\bar{\rho}} \to 0$

$$\Rightarrow \rho_E : \Pi \to \underset{\substack{\cup\!\cup \\ GL_{2n}(K)}}{GL_n(K[\varepsilon])} \text{ is given by } \begin{pmatrix} \bar{\rho}(g) & A_g \\ 0 & \bar{\rho}(g) \end{pmatrix} = M\rho'_E(g). \quad \#$$