Obstructed and unobstructed deformation problems

Π profinite, w/ Φ_p
$\bar\rho : \Pi \to GL_n(k)$, $C(\bar\rho) = k$.

Recall:

$$\mathrm{Hom}_k\left({}^{m_{R_{\bar\rho}}}\!/\!_{(m^2_{R_{\bar\rho}}, m_\Lambda)}, k \right) \cong \mathrm{Hom}_\Lambda(R_{\bar\rho}, k[\varepsilon]) = \underbrace{D_{\bar\rho,\Lambda}(k[\varepsilon])}_{= t_{D_{\bar\rho,\Lambda}}} \cong H^1(\Pi, \mathrm{Ad}(\bar\rho)) \quad \mathrm{Ad}(\bar\rho) = M_n(k) \circlearrowleft \Pi$$

$$\cong \mathrm{Ext}^1_{k[[\Pi]]}(V_{\bar\rho}, V_{\bar\rho}), \quad V_{\bar\rho} \cong k^n \circlearrowleft \Pi.$$

$d_1 := \dim_k(D_{\bar\rho,\Lambda}(k[\varepsilon]))$.

We try to compute the obstruction to lift a homomorphism, i.e. given $R_1, R_0 \in \mathcal{C}_\Lambda$ and $\varphi : R_1 \to R_0 \to 0$, $I := \mathrm{Ker}(\varphi)$ s.t. $m_{R_1} I = (0)$ so that I is a $\underbrace{R_1/m_{R_1}}_{= k}$-e.v. (e.g. φ small).

Given $\rho_0 : \Pi \to GL_n(R_0)$, what keeps us from finding a lift to R_1?

$$\begin{tikzcd} \exists\, \gamma : \Pi \arrow[r] \arrow[dr, "\rho_0"'] & GL_n(R_1) \arrow[d, "\varphi"] \\ & GL_n(R_0) \end{tikzcd} \quad \text{just a map.}$$

γ is homomorphism if $\forall g_1, g_2 \in \Pi$, $c_\gamma(g_1, g_2) = \gamma(g_1 g_2)\gamma(g_2)^{-1}\gamma(g_1)^{-1} = Id$

Notice, $c_\gamma(g_1, g_2) = Id + d_\gamma(g_1, g_2)$, $d_\gamma(g_1, g_2) \in M_n(I) = \mathrm{Ad}(\bar\rho) \otimes I$.

<u>Check</u>:

Can assume φ small

$d_\gamma : \Pi \times \Pi \to \mathrm{Ad}(\bar\rho) \otimes I$ is a 2-cocycle. (why?)

<u>Rec</u>: A 2-cocycle is $f : \Pi \times \Pi \to M \circlearrowleft \Pi$ s.t. $f(g,1) = f(1,g)$ and

$$g \cdot f(h, l) - f(gh, l) + f(g, hl) - f(g, h) = 0$$

A 2-coboundary is f is $\exists\, h : \Pi \to M \mid f(g_1, g_2) = g_1 h(g_2) - h(g_1 g_2) + h(g_1) = \delta h$

If γ' is another set-theoretical lift $\Rightarrow d_{\gamma'} = d_\gamma + b$, b coboundary

(check) $\Rightarrow [\gamma] \equiv$ class of set-theoretical lifts of $\rho_0 \equiv \mathcal{O}(\rho_0) \in H^2(\Pi, Ad(\bar{\rho}) \otimes I)$.

Moreover, $\mathcal{O}(\rho_0)$ is trivial if $\exists\, \gamma$, homomorphism lifting ρ_0.

<u>Def</u>: $\mathcal{O}(\rho_0) \equiv$ obstruction class of ρ_0 relative to $\varphi: R_1 \to R_0$.

If $H^2(\Pi, Ad(\bar{\rho})) = 0 \Rightarrow$ the deformation problem is particularly simple.

<u>Thm 4.2. (Mazur)</u> Supp $C(\bar{\rho}) = k$, $d_1 := \dim H^1(\Pi, Ad(\bar{\rho}))$, $d_2 := \dim H^2(\Pi, Ad(\bar{\rho}))$

$\Rightarrow$ Krulldim $(R_{\bar{\rho}}/m_\Lambda R_{\bar{\rho}}) \geq d_1 - d_2$. $(**)$

If $d_2 = 0 \Rightarrow (**)$ is equality and $R_{\bar{\rho}} \cong \Lambda[[T_1, \dots, T_{d_1}]]$.

<u>Proof</u>

$\exists \quad \Lambda[[T_1, \dots, T_{d_1}]] \to R_{\bar{\rho}} \to 0$ inducing an isomorph. on tangent spaces

unique max: $(T_1, \dots, T_{d_1}, p)_\Lambda$

$\to$ tangent $(T_1, \dots, T_{d_1}, p)_\Lambda / (T_1, \dots, T_{d_1}, p)^2_\Lambda + m_\Lambda$, $m_\Lambda = (p)$

has dim d_1 as k-vector space.

same as ~~$R_{\bar\rho}$~~ $m_{R_{\bar{\rho}}}/m^2_{R_{\bar{\rho}}} + m_\Lambda$.

dim d_1 as well.

Reducing modulo maximal ideals: $\to$ tgt is $m_R/m_R^2 + m_\Lambda$: $\dfrac{m_\Lambda m_R}{m_\Lambda^2 m_R^2 + m_\Lambda^2}$

$k[[T_1, \dots, T_d]] \to R_{\bar{\rho}}/m_\Lambda R_{\bar{\rho}} \to 0$ $\qquad (m_R/m_\Lambda) / (m_R/m_\Lambda)^2$

also induces isomorphism on tgt spaces

$F := k[[T_1, \dots, T_{d_1}]]$, $m_F = (T_1, \dots, T_{d_1})$.

$0 \to J \to F \to R_{\bar\rho}/m_\Lambda R_{\bar\rho} \to 0$ i.e. $F/J \cong R_{\bar\rho}/m_\Lambda R_{\bar\rho}$ as vector spaces.

$$\text{Krulldim}(R_{\bar\rho}/m_\Lambda R_{\bar\rho}) = \dim_k(R_{\bar\rho}/m_\Lambda R_{\bar\rho}) = \underbrace{\dim_k(F)}_{= d_1} - \dim_k(J)$$

Need to check that J has at most d_2 generators $\Rightarrow \dim \leq d_2$.

Since $m_F J \subseteq J$, consider

$$0 \to J/m_F J \to F/m_F J \to R_{\bar\rho}/m_\Lambda R_{\bar\rho} \to 0$$

$\uparrow$ this still induces isomorphism on tgt spaces.

$$\Rightarrow \dim_k(R_{\bar\rho}/m_\Lambda R_{\bar\rho}) \geq d_1 - \dim_k(J/m_F J).$$

Must check $\dim_k(J/m_F J) \leq d_2$:

Let ρ_p = image of $\rho_{univ}: \Pi \to GL_n(R_{\bar\rho})$

$$\begin{tikzcd} \Pi \arrow[r] \arrow[dr, "\rho_p"'] & GL_n(R_{\bar\rho}) \arrow[d] \\ & GL_n(R_{\bar\rho}/m_\Lambda R_{\bar\rho}) \end{tikzcd}$$

This is universal among defs of $\bar\rho$ to Λ-algebras killed by m_Λ (i.e. k-algebras).

$\Rightarrow$ This gives $O(\rho_p) \in H^2(\Pi, Ad(\bar\rho)) \otimes J/m_F J$,

obstruction to lift ρ_p to $F/m_F J$.

Consider:

$$f: J/\mathfrak{m}_F J \to k.$$

$$\mathrm{Hom}_k(J/\mathfrak{m}_F J, k) \xrightarrow{\alpha} H^2(\Pi, Ad(\bar\rho))$$

$$f \longmapsto (1\otimes f)\,\Theta(\mathbb{P}_\rho) \in \cdots \otimes k$$

$$\overset{\shortparallel}{(\eta_2 \otimes J/\mathfrak{m}_F J)}$$

We claim α is injective, which is enough:

Let $f \neq 0$, $f \in \mathrm{Ker}(\alpha)$. Let $A = (F/\mathfrak{m}_F J)/\mathrm{Ker}(f)$

Let I = image in A of $J/\mathfrak{m}_F J =$

$= (J/\mathfrak{m}_F J)/\mathrm{Ker}(f) = \mathrm{Im}(f) = k.$

$$\Rightarrow 0 \to \underset{\underset{k}{\wr\!\wr}}{I} \to A \to R_{\bar\rho}/\mathfrak{m}_\Lambda R_{\bar\rho} \to 0.$$

induces isomorp. on tgt. spaces.

Now $\underset{\underset{(1\otimes f)(\Theta(\mathbb{P}_\rho))}{\shortparallel}}{\Theta(\mathbb{P}_\rho) = 0} \Rightarrow \exists$ def of $\bar\rho$ to A lifting $\mathbb{P}_\rho$, say ρ_1

But A k-algebra, $\mathbb{P}_\rho$ universal among such rings $\Rightarrow$

$\exists\ R_{\bar\rho}/\mathfrak{m}_\Lambda R_{\bar\rho} \xrightarrow{\xi} A$ s.t. $\xi \circ \mathbb{P}_\rho = \rho_1$

$$0 \to I \to A \to R_{\bar\rho}/\mathfrak{m}_\Lambda R_{\bar\rho} \to 0 \quad \Rightarrow A \simeq I \oplus R_{\bar\rho}/\mathfrak{m}_\Lambda R_{\bar\rho} \quad !!$$

dim d_1

d_1 $\quad$ k

Finally, $d_2 = 0 \Rightarrow \Lambda[[T_1,\dots,T_d]] \xrightarrow{\sim} R_{\bar\rho} \to 0.$ #

Re-interpretation:

K number field, S finite set of places of K, including all above p and the archimedean ones. $S_\infty \subseteq S$ = set of archimedeans.

$\Pi := G_{K,S}$; $\bar{\rho}: \Pi \to GL_n(\bar{k})$ | $c(\bar{\rho}) = k$.

<u>Tate's global Euler char formula:</u> ($K/\mathbb{Q}$ deg d, M $G_{K,S}$-mod, finite, S contains $\ell \mid |M|$).

$$\frac{|H^0(G_{K,S}, M)|\,|H^2(G_{K,S}, M)|}{|H^1(G_{K,S}, M)|} = \frac{1}{|M|^d} \prod_{v \in S_\infty} |H^0(G_{K_v}, M)|$$

For us, $M = \mathrm{Ad}(\bar{\rho})$, k-vector space $\Rightarrow$ order is p-power $\Rightarrow$ all orders are p-powers and hence

$$\dim H^0(G_{K,S}, M) + \dim H^1(G_{K,S}, M) + \dim H^2(G_{K,S}, M) = \sum_{v \in S_\infty} \dim H^0(G_{K_v}, M) - d \dim M$$

$$\Rightarrow d_0 - d_1 + d_2 = \sum_{v \in S_\infty} \dim H^0(G_{K_v}, \mathrm{Ad}(\bar{\rho})) - dn^2$$

$$d_0 = \dim H^0(G_{K,S}, \mathrm{Ad}(\bar{\rho})) \underset{\dim}{=} \mathrm{Ad}(\bar{\rho})^{G_{K,S}} \overset{c(\bar{\rho}) = k}{=} 1$$

$$\Rightarrow \text{Krull dim}(R_{\bar{\rho}}/m_\Lambda R_{\bar{\rho}}) \geq d_1 - d_2 = \boxed{1 + dn^2 - \sum_{v \in S_\infty} \dim H^0(G_{K_v}, \mathrm{Ad}(\bar{\rho}))}$$