Chapter 5: Deformations with prescribed properties

supp $C(\bar\rho) = k \Rightarrow \exists R_{\bar\rho}$ universal def. ring, $\mathrm{Specf}(R_{\bar\rho})$ ($\leftarrow$ think of it as $\mathrm{Spec}(R_{\bar\rho})$!) = deformation space, i.e. $\forall P \in \mathrm{Specf}(R_{\bar\rho})$, $P \leftrightarrow R_{\bar\rho} \xrightarrow{\pi_P} R_{\bar\rho}/P$ (prime in $R_{\bar\rho}$)

$\Rightarrow \exists!$ deformation of $\bar\rho$ to $R_{\bar\rho}/P$.

e.g. (chapter 5) $R_{\bar\rho} = \mathbb{Z}_p[[T_1, T_2, T_3]]$ (i.e. unobstructed) 3-dimensional / $\mathbb{Z}_p$,

T_i = coordinates : $\forall (m_1, m_2, m_3)$ in the max ideal of some $\mathbb{Z}_p$-algebra, map $T_i \mapsto m_i$ we get a deformation.

But in many circumstances we want only ρ, def of $\bar\rho$ satisfying certain conditions. The universal def ring for these will be a quotient of $R_{\bar\rho}$.

• Deformation conditions (Mazur)

want: $R \rightsquigarrow \{$deformations of $\bar\rho$ satisfying our condition$\}$ to be a subfunctor of $D_{\bar\rho}$.

Want: this subfunctor to be relatively representable.

<u>Recall</u>: $\rho: \Pi \to GL_n(A)$ rep $\Leftrightarrow$ $\Pi \times A^n \mapsto A^n$ $\leftarrow$ after fixing a basis of A^n.

$(\sigma, v) \mapsto \rho(\sigma)v$

<u>Def</u>: A, A_1 artinian coeff Λ-algebras. Given $\rho: \Pi \to GL_n(A)$ and $\alpha: A \to A_1$ morphism of coeff Λ-alg. $\rightsquigarrow \rho_1: \Pi \to GL_n(A) \xrightarrow{\alpha} GL_n(A_1)$

we say that ρ_1 $(=: \alpha_* \rho)$ is the pushforward of ρ by α

$\alpha_* = D_{\bar\rho}(\alpha)$

<u>notice</u>: α_* is $\otimes_A A_1$.

<u>Def</u>: Let $\bar\rho : \Pi \to GL_n(K)$ a residual rep. A <u>deformation condition</u> on deformations of $\bar\rho$ is a property Q of n-dimensional reps. of Π defined over artinian coeff Λ-algebras s.t.

for subfunctor not to be trivial. $D_{\bar\rho}(K) = \{*\}$

i) $\bar\rho$ has property Q. (K is artinian, local, complete)

ii) Given $\rho : \Pi \to GL_n(A)$ def of $\bar\rho$ and $\alpha : A \to A_1$, if ρ has property $Q \Rightarrow \alpha_*\rho$ has property Q. ← gives a functor.

iii) Let

$$\begin{array}{ccc} & A\times_C B & \\ p \swarrow & & \searrow q \\ A & & B \\ \alpha \searrow & & \swarrow \beta \\ & C & \end{array}$$

be a fiber product diagram and $\rho : \Pi \to GL_n(A\times_C B)$ def of $\bar\rho$.

↓ schlessinger criteria. Rel. rep.

then ρ has prop $Q \iff p_*\rho, q_*\rho$ have property Q.

⇓

iv) $\alpha : A \hookrightarrow A_1$ inj. hom of coeff Λ-alg., $\rho : \Pi \to GL_n(A)$ def of $\bar\rho$. If $\alpha_*\rho$ has prop $Q \Rightarrow \rho$ has so.

<u>Def</u>: Q def. condition for $\bar\rho$. Define $D_{\bar\rho, Q} : \mathcal{C}^0_\Lambda \to Sets$

$D_{\bar\rho, Q}(A) = \{$ defs of $\bar\rho$ to A w/ prop. $Q \}$.

$D_{Q,\bar\rho}(R) := \varprojlim_k D_{Q,\bar\rho}(R/m^k) \Rightarrow$ extend $D_{\bar\rho,Q} : \mathcal{C}_\Lambda \to Sets$.

Namely: a def of $\bar{\rho}$ to a Λ-algebra R has condition $Q \Leftrightarrow$ its reductions modulo m^k have so $\forall k$.

P.6.1 $D_{\bar{\rho},Q}$ is a subfunctor of $D_{\bar{\rho}}$.

Thm 6.1 If Q is a deformation condition for $\bar{\rho} \Rightarrow D_{\bar{\rho},Q}$ satisfies H1, H2, H3 in Schlessinger's thm. ($\Rightarrow$ frames are rep by $R^{\square}_{\bar{\rho},Q} = R^{\square}_{\bar{\rho}}/P$).

If $C(\bar{\rho}) = k \Rightarrow D_{\bar{\rho},Q}$ satisfies H4 $\Rightarrow$ it's rep by $R_{\bar{\rho},Q} = R_{\bar{\rho}}/P$.

Supp Q def. condition. Consider $D_{\bar{\rho},Q}(k[\varepsilon]) \subseteq D_{\bar{\rho}}(k[\varepsilon]) \simeq H^1(\Pi, Ad(\bar{\rho}))$

Def: $H^1_Q(\Pi, Ad(\bar{\rho}))$ = subspace of $H^1(\Pi, Ad(\bar{\rho}))$ corresponding to $D_{\bar{\rho},Q}(k[\varepsilon])$.

- Deformations w/ fixed determinant (most natural condition to impose)

Representations attached to elliptic curves have determinant equal to the cyclo char. (this will be asked in the last assignment)

Def: Let $\delta : \Pi \to \Lambda^*$ be a cont homomorphism i.e. a character, and $\forall$ coeff Λ-algebra R, $\delta_R : \Pi \xrightarrow{\delta} \Lambda^* \to R^*$.

We say that a def. ρ of $\bar{\rho}$ to R has determinant δ if $\det(\rho) = \delta_R = i \circ \delta$.

$i : \Lambda \to R$

Lemma 6.2. Supposed $\det(\bar{\rho}) = \delta_k$ i.e. has determinant $\delta \Rightarrow$ "$\det = \delta$" is a def. condition (P, 6.4)

Suppose $C(\bar{\rho}) = K \Rightarrow \exists\, R_{\bar{\rho}}^{\det=\delta}$ universal def ring corresponding to this condition.

Q. Tangent space?

$Ad^0(\bar{\rho}) = \{\gamma \in M_n(K)$ s.t $tr(\gamma) = 0\} \circlearrowleft \Pi$ check!

<u>Lemma</u>: If $p \nmid n \Rightarrow D_{\det=\delta}(K[\varepsilon]) = H^1(\Pi, Ad^0(\bar{\rho})) \subseteq H^1(\Pi, Ad(\bar{\rho}))$

If $p | n \Rightarrow H^1(\Pi, Ad^0(\bar{\rho})) \not\subseteq H^1(\Pi, Ad(\bar{\rho}))$ but $Ad^0(\bar{\rho}) \hookrightarrow Ad(\bar{\rho})$

still induces $D_{\det=\delta}(K[\varepsilon]) = Im[H^1(\Pi, Ad^0(\bar{\rho})) \to H^1(\Pi, Ad(\bar{\rho}))]$.

<u>Proof</u>: Recall that lifts to $K[\varepsilon]$ give raise to cohomology classes. Observe:

$$1+\varepsilon b \longmapsto b$$

$$\begin{array}{ccc} 1+\varepsilon M_n(K) & \longrightarrow & M_n(K) \\ \downarrow \det & \circlearrowleft & \downarrow Tr \\ 1+\varepsilon K & \longrightarrow & K \end{array}$$

The requirement of fixed det forces the $1+\varepsilon M_n(K)$ part of the lift to have det $1 \Rightarrow b$ should have trace zero.

$\Rightarrow$ we get elements in $H^1(\Pi, Ad(\bar{\rho}))$ represented by cocycles taking values in $Ad^0(\bar{\rho}) \Rightarrow \in H^1(\Pi, Ad^0(\bar{\rho}))$. If $p \nmid n$ the image is itself. #

$$\begin{array}{c} H^1(\Pi, Ad^0(\bar{\rho})) \\ \cap \\ H^1(\Pi, Ad(\bar{\rho})) \end{array}$$

P.6.5. $C(\bar\rho)=\kappa$, $\det(\bar\rho)=\delta$.

$R = R_{\bar\rho}$, $\mathbb{P} = \rho_{univ}$. $R^{\delta}_{\bar\rho}$ = univ def ring of det δ.

$$\delta_R : \Pi \xrightarrow{\delta} \Lambda^* \xrightarrow{\pi} R^*_{\bar\rho}$$

$$\Rightarrow R^{\delta}_{\bar\rho} = R_{\bar\rho} / \overline{\langle \delta_{R_{\bar\rho}}(g) - \det \mathbb{P}(g) \rangle}$$

Obs : $p \nmid n$, $C(\bar\rho)=\kappa \Rightarrow R_{\bar\rho} = R^{\delta}_{\bar\rho} \hat\otimes_{\Lambda} \Lambda[[\Gamma]]$

- Categorical deformation conditions (Ramakrishna)

idea : require that our (artinian) def's define Π-modules belonging to certain subcategory of Λ-modules of finite length.

i.e. $\sigma(\lambda v)=\lambda\sigma(v)$

$\mathcal{P}$ = subcategory of Λ-modules of finite length w/ cont Λ-linear Π-action.

(e.g. $\mathcal{P}$ abelian category)

Assume $\mathcal{P}$ closed by sub-objects, quotients and finite $\oplus$.

Def

Given $\bar\rho$, a def. $\rho : \Pi \to GL_n(R)$, is of type $\mathcal{P}$ if the Π-modules defined by $\rho_k : \Pi \to GL_n(R/m^k)$ are in $\mathcal{P}$.

i.e. $(R/m^k)^n \circlearrowleft \Pi$

Thm (Ramakrishna) If $\bar\rho$ is of type $\mathcal{P} \Rightarrow$ the condition "being of type $\mathcal{P}$" is a def condition. (self study) e.g.

$\Pi = G_{\mathbb{Q}_\ell}$, $\mathcal{P} = \mathcal{P}_{fl}$ = $G_{\mathbb{Q}_\ell}$-reps ρ over artinian A s.t A^n isomorphic as $G_{\mathbb{Q}_\ell}$-module to the generic fiber of a finite flat group scheme over $\mathrm{Spec}(\mathbb{Z}_\ell)$ $\rightsquigarrow$ ~~$\mathrm{Spec}(\mathbb{Z}_\ell[x,y]/(y^2-x^3-ax-b))$~~ $E/_{\mathbb{Z}_\ell}[n]$

$\downarrow$ $\leftarrow$ finite flat morphism.

$\mathrm{Spec}(\mathbb{Z}_\ell) = \{(0), (\ell)\}$

$\uparrow$ generic.

deformations satisfying $\mathcal{P}_{fl}$ are called flat deformations.

K is artinian. (domain).

$I_1 \supseteq \ldots$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{array}{ccc} & & GL_n(\Lambda^*) \\ & \overset{\delta}{\nearrow} & \downarrow \pi \\ \bar{\rho} : \Pi & \longrightarrow & GL_n(K) \end{array}$$

$\det(\bar{\rho}) = \pi \circ \delta$

have $\det \delta : \Pi \to \Lambda^*$

$\rho : \Pi \to GL_n(K[\varepsilon])$

$$\begin{array}{ccc} & \overset{\delta}{\nearrow} & GL_n(\Lambda^*) \\ \Pi & & \downarrow \pi \\ & \underset{\rho}{\searrow} & GL_n(K) \subseteq GL_n(K[\varepsilon]) \end{array}$$

$\det(\rho) = \pi \circ \delta$

$\rho : \Pi \to GL_n(K[\varepsilon]) = K^\times (1 + \varepsilon M_n(K))$.

$\det(\rho) = \det(\bar{\rho}) \Rightarrow 1 + \varepsilon K_0 = 1$