Math Camp - Analysis

Preliminaries

Our goal:

$$\max_{x \in X} f(x)$$
$$\text{s.t. } g(x) \geq 0$$

We want to really understand this problem

Sets

X is a set. A set is a collection of stuff

Sets

X is a set. A set is a collection of stuff

- $X = \{\text{Apples, Bananas, Oranges}\}$
- $X = \{x : x \text{ is a fruit}\}$
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ (set of natural numbers)
- $X = \{x \in \mathbb{N} : x/2 \in \mathbb{N}\}$ (set of even natural numbers)
- $\mathbb{Q} = \{x : x = a/b \, a, b \in \mathbb{Z}, \, b > 0\}$ (set of rational numbers)
- $\mathbb{R}$ set of real numbers

Our basic building block. Usually denoted by capital letters like A, B, C, X, Y.

Sets

Some notation

- $x \in X$, x is an element of set X.

Sets

Some notation

- $x \in X$, x is an element of set X.
- $Y \subseteq X$, Y is a subset of X
    - If $x \in Y$ then $x \in X$.

Sets

Some notation

- $x \in X$, x is an element of set X.
- $Y \subseteq X$, Y is a subset of X
    - If $x \in Y$ then $x \in X$.
- $X \cup Y$ is the union of X and Y, the set of all elements contained in X or Y.

Sets

Some notation

- $x \in X$, x is an element of set X.
- $Y \subseteq X$, Y is a subset of X
  - If $x \in Y$ then $x \in X$.
- $X \cup Y$ is the union of X and Y, the set of all elements contained in X or Y.
- $X \cap Y$ is the intersection of X and Y. The set of all elements contained in both X and Y.

Sets

Some notation

- $x \in X$, x is an element of set X.
- $Y \subseteq X$, Y is a subset of X
  - If $x \in Y$ then $x \in X$.
- $X \cup Y$ is the union of X and Y, the set of all elements contained in X or Y.
- $X \cap Y$ is the intersection of X and Y. The set of all elements contained in both X and Y.
- $X \setminus Y$ is the set of all elements in X that aren't in Y.

Sets

Some notation

- $x \in X$, x is an element of set X.
- $Y \subseteq X$, Y is a subset of X
  - If $x \in Y$ then $x \in X$.
- $X \cup Y$ is the union of X and Y, the set of all elements contained in X or Y.
- $X \cap Y$ is the intersection of X and Y. The set of all elements contained in both X and Y.
- $X \setminus Y$ is the set of all elements in X that aren't in Y.
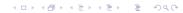- $\emptyset$ is the empty set, a set with no elements

Sets

Some notation

- $x \in X$, x is an element of set X.
- $Y \subseteq X$, Y is a subset of X
  - If $x \in Y$ then $x \in X$.
- $X \cup Y$ is the union of X and Y, the set of all elements contained in X or Y.
- $X \cap Y$ is the intersection of X and Y. The set of all elements contained in both X and Y.
- $X \setminus Y$ is the set of all elements in X that aren't in Y.
- $\emptyset$ is the empty set, a set with no elements
- $P(X)$ (sometimes 2^X), the set of all subsets of X

  $P(\{Apples, Bananas\}) = \{\{Apples\}, \{Bananas\}, \emptyset, \{Apples, Bananas\}\}$

  (if a set has k elements, the power set has 2^k elements)

- $X \times Y$, the set of ordered pairs (x, y)

Real numbers

We are mostly going to work with the set $\mathbb{R}$. This is the set of all numbers you should be used to dealing with.

An important property of $\mathbb{R}$ is the least upper bound property.

Definition (Least upper bound)

Any set $X \subseteq R$ with an upper bound, has a least upper bound,

$$c = \sup X$$

i.e. $c \geq x$ for all $x \in X$ and if $d > x$ for all $x \in X$ then $c \leq d$.

A lot of what we are going to do in this class is finding conditions for when $\sup X \in X$. This is clearly true for some sets (like $[0, 1]$) and not true for others (like $[0, 1)$).

Functions

$$f : X \to Y$$

- ▶ Takes each element from X (the domain), maps it to an element of Y (the range).
- ▶ $f(x)$ is value of the function f when evaluated at $x \in X$.

Functions

$$f : X \rightarrow Y$$

- ▶ Takes each element from X (the domain), maps it to an element of Y (the range).
- ▶ $f(x)$ is value of the function f when evaluated at $x \in X$.
- ▶ Examples:
    - ▶ Utility function $u : \{Apples, Bananas\} \rightarrow \mathbb{R}$ - how much utility you get from consuming Apples or Bananas
    - ▶ Production function $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, $f(L, K) = L^\alpha K^{1-\alpha}$ - how much you produce when you work L hours with K units of capital.
    - ▶ Choices: $C : P(\{Apples, Bananas, Oranges\}) \setminus \emptyset \rightarrow \{Apples, Bananas, Oranges\}$, what you would choose from each possible set of goods

Linear functions

A linear function $f : \mathbb{R}^m \to \mathbb{R}^n$ is a function such that

- $f(x + y) = f(x) + f(y)$
- For scalar a, $f(ax) = af(x)$.

Every linear function can be expressed as $f(x) = Ax$ for some $A \in \mathbb{R}^{n \times m}$.

A function is affine linear if it is of the form $Ax + b$ for some $b \in \mathbb{R}^n$

Linear functions

Even though a affine linear function is defined over all of $\mathbb{R}$, we only need to know the value of it at a few points to know everything about it.

Makes a lot of things straightforward

- ▶ Finding solutions to $Ax = b$. We know from linear algebra we can solve this with Gaussian elimination. We also know when a solution exists and when it's unique.
- ▶ Finding the direction the function is increasing in.
- ▶ Finding where linear functions intersect
- ▶ ect.

Unfortunately, we want to work with non-linear functions. Can we define a rich class of functions that sort of work like linear functions?

Our Goal

Theorem (The Extreme Value "Theorem")

*A function $f : X \to \mathbb{R}$ that satisfies **???** has a maximum.*

- ▶ Every (bounded) function has a supremum (i.e. $\sup f(X)$)
- ▶ Clearly not every function has a maximum.
    - ▶ $f(x) = x$ on $\mathbb{N}$, $\mathbb{R}$, $[a, b)$...
- ▶ Let's think a bit about how to prove this.

Extreme Value Theorem

Theorem (The Extreme Value "Theorem')

*A function $f : X \to \mathbb{R}$ that satisfies **???** has a maximum.*

How might we show this?

Extreme Value Theorem

Theorem (The Extreme Value "Theorem')

*A function $f : X \to \mathbb{R}$ that satisfies **???** has a maximum.*

How might we show this?

1. We know $c = \sup_{x \in X} f(x)$ exists.

Extreme Value Theorem

Theorem (The Extreme Value "Theorem')

*A function $f : X \to \mathbb{R}$ that satisfies **???** has a maximum.*

How might we show this?

1. We know $c = \sup_{x \in X} f(x)$ exists.
2. For any $a < c$, we know there's an x'_a such that $f(x'_a) \in (a, c]$.

Extreme Value Theorem

Theorem (The Extreme Value "Theorem')

*A function $f : X \to \mathbb{R}$ that satisfies **???** has a maximum.*

How might we show this?

1. We know $c = \sup_{x \in X} f(x)$ exists.
2. For any $a < c$, we know there's an x_a' such that $f(x_a') \in (a, c]$.
3. Conjecture: $x_n = x_{c-1/n}'$ gets arbitrarily close to the maximizer.

Vectors

Consider two vectors $x, y \in \mathbb{R}^n$.

How do we measure distance. If $n = 1$ it's obvious

- The length of x is just $|x|$, and the distance between x, y is $|x - y|$.
- We can do the same thing in $\mathbb{R}^n$. The length of a vector is measured using the norm

$$||x|| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

and the distance between two vectors (called a metric) is $d(x, y) = ||x - y||$.

Distances

These satisfy the properties we might expect a distance to satisfy:

1. $||x|| = 0$ iff $x = 0$.
2. $||ax|| = |a| \cdot ||x||$ for every $a \in \mathbb{R}$.
3. $||x - y|| = ||y - x||$.
4. $||x + y|| \leq ||x|| + ||y||$ (triangle inequality)
5. $|x \cdot y| \leq ||x|| ||y||$ (Cauchy-Schwarz inequality)

(There are other functions that satisfy 1-4. At least in $\mathbb{R}^n$, these are equivalent in terms of defining continuity, open sets, etc. It is occasionally more convenient to work with $||x||_\infty = \max |x_i|$ or $||x||_1 = \sum_{i=1}^{n} |x_i|$.)

Open sets

With our distance, we can define the set of close together points

Definition (Neighborhood)

For any $x \in \mathbb{R}^n$ and $\epsilon > 0$ the open neighborhood $B_\epsilon(x)$ with center x and radius ϵ is defined to be the set of all $y \in \mathbb{R}^n$ s.t. $||y - x|| < \epsilon$.

Open sets

With our distance, we can define the set of close together points

Definition (Neighborhood)

For any $x \in \mathbb{R}^n$ and $\epsilon > 0$ the open neighborhood $B_\epsilon(x)$ with center x and radius ϵ is defined to be the set of all $y \in \mathbb{R}^n$ s.t. $||y - x|| < \epsilon$.

In $\mathbb{R}$, these are open intervals centered around x. In $\mathbb{R}^n$, these are balls with center x

Open sets

Using this, we define a bunch of "nice" sets

- ▶ x is a limit point of $E \subseteq \mathbb{R}^n$ if every nbhd of x contains a $y \in E$, $y \neq x$.
- ▶ A set E is closed if E contains all its limit points.
- ▶ A point x is an interior point of E if there exists some $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq E$.
- ▶ A set is open if every point is interior.
- ▶ A set is bounded if there is a $M \in \mathbb{R}$ s.t. $||x|| < M$ for all $x \in E$.

A set $E \subseteq \mathbb{R}^n$ is open iff the complement of $E^c := \mathbb{R}^n \setminus E$ is closed.

Some examples

Verify these on your own

- (a, b) is open.
- $[a, b]$ is closed
- $\{1, 1/2, 1/3, 1/4, \ldots\}$ is neither open or closed.
- $\emptyset$ is open and closed
- $\mathbb{R}^n$ is open and closed.
- Unions and finite intersections of open sets are open.

Sequences

When thinking about these nonlinear things, its often very hard to solve problems directly

- ▶ How do you find the area under a curve?
- ▶ How do I find a root?
- ▶ How do I find the largest value of a function?

But we can often define a sequence of easier problems that get closer and closer to the problem we want to solve

Think about our x_n's, $f(x_n)$ is getting arbitarily close to the supremum of $f(\cdot)$.

Sequences

A sequence $\{x_n\}$ is an ordered list of numbers. What happens to this in the long run?

We say a sequence converges if there exists an $x^* \in \mathbb{R}^m$ s.t. for every $\epsilon > 0$ there exists an N s.t. if $n \geq N$ $||x_n - x^*|| < \epsilon$

If a sequence converges to x^*, we write $\lim_{n\to\infty} x_n = x^*$ or $x_n \to x^*$.

Sequences

Sequences may or may not converge

- $x_n = 1$ converges to 1
- $x_n = n$ diverges (goes to $+\infty$)
- $x_n = 1/n$ converges to 0.
    - $|0 - \frac{1}{n}| < \epsilon$ if $n \geq 1/\epsilon$.
- $x_n = (-1)^n$ does not converge or diverge.

Convergence

Theorem

Let $\{x_n\}$ be a sequence in $\mathbb{R}^n$.

- $\{x_n\}$ converges to $x \in \mathbb{R}^n$ iff every nbhd of x contains all but finitely many terms of $\{x_n\}$.
- If $\{x_n\}$ converges to both x and x' then $x = x'$.
- If $\{x_n\}$ converges then it is bounded.
- If $x \in E$ and x is a limit point of E then there is a sequence $\{x_n\}$ in E s.t. $\lim x_n = x$.
- A sequence converges iff each of it's components converge, e.g. $x_n = (x_{1,n}, x_{2,n}, \ldots x_{k,n}) \to x = (x_1, x_2, \ldots x_k) \iff x_{i,n} \to x_i$ for all $i \in \{1, \ldots k\}$.

Convergence

Convergence tells us that eventually all points in the sequence approximate the limit point.

Another natural convergence criteria would be the all points in the sequence eventually behave like each other.

Definition

A sequence is Cauchy if for any $\epsilon > 0$ there exists an N s.t. if $n, m \geq N$ s.t. $|x_n - x_m| < \epsilon$

Every convergent sequence is Cauchy, and every Cauchy sequence converges in $\mathbb{R}$. The second part of that statement is not true in general (think about $1/n$ as a sequence in $(0, 1]$ or $x_n = \sqrt{2}$ up to the nth decimal place as a sequence in $\mathbb{Q}$.

Subsequences

Given a sequence $\{x_n\}$, we can define an infinte sequence of positive intergers $\{n_k\}$, $n_1 < n_2 < n_3 \ldots$. The sequence $\{x_{n_k}\}$ is a subsequence. If the sequence x_n converges to x, then so does the subsequence.

Subsequences

Given a sequence $\{x_n\}$, we can define an infinte sequence of positive intergers $\{n_k\}$, $n_1 < n_2 < n_3 \ldots$. The sequence $\{x_{n_k}\}$ is a subsequence. If the sequence x_n converges to x, then so does the subsequence.

Who cares?

Subsequences

Given a sequence $\{x_n\}$, we can define an infinte sequence of positive intergers $\{n_k\}$, $n_1 < n_2 < n_3 \ldots$. The sequence $\{x_{n_k}\}$ is a subsequence. If the sequence x_n converges to x, then so does the subsequence.

Who cares?

Even if a sequence doesn't converge, maybe we can find a subsequence that does.

Compactness

Definition
We say a set C is (sequentially) compact if every sequence that lies in C has a convergent subsequence.

- Asking for a set where every sequence converges is way too strong (why?)
- This is a very nice, strong property.
- Apriori, hard to see if a set satisfies it or not.
- Clearly, compact sets must be closed. What else do we need?

Bolzano Weierstrass

Theorem

Any bounded sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$

- ▶ Consider the set $I_1 = [\inf x_n, \sup x_n]$.
- ▶ Cut it in half, one half must have infinitely many points. Let that half be I_2
- ▶ Repeat

Then let x_{n_1} be a point in I_1, x_{n_2} be a point in I_2 s.t. $n_2 > n_1$ and so on. Clearly cauchy so converges.

Compactness

- ▶ Previous theorem tells compact if closed and bounded.
- ▶ Reverse is also true, closed and bounded if compact.
  - ▶ This is straightforward to show, if the set is either not closed or not bounded can you construct a sequence with no convergent subsequence?

Functions in $\mathbb{R}$

We are mostly going to work with functions that are defined on some "nice" subset of $\mathbb{R}$

- ▶ Lots of our problems are of this form (utility maximization, the firm's problem, econometrics)
- ▶ For finite sets, our maximization problem is "easy," just check ever value
- ▶ For infinite sets, our problem seems really hard. $\mathbb{R}$ is a natural set that's "rich" enough to let us do things
    - ▶ Think about questions like "where do 2 lines intersect?," is that easier to solve in $\mathbb{Q}$ or $\mathbb{R}$.

Continuity

Back to our theorem.

Theorem (The Extreme Value "Theorem")

*A function $f : X \to \mathbb{R}$ that satisfies **???** has a maximum.*

- ▶ Compactness seems like a promising property for X. Then we have a subsequence s.t. $x_{n_k} \to x^* \in X$.
- ▶ Recall $\sup_{x' \in X} f(x') - f(x_{n_k}) < 1/n_k$.
- ▶ So x seems like a natural choice for our maximum

Continuity

Immediately we see our first problem:

- ▶ Functions in $\mathbb{R}$ can be really messy.
- ▶ Need more structure
- ▶ Natural bit of structure: Nearby points should "behave similarly"

Continuity

Immediately we see our first problem:

- ▶ Functions in $\mathbb{R}$ can be really messy.
- ▶ Need more structure
- ▶ Natural bit of structure: Nearby points should "behave similarly"

Definition (Continuity)

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x' \in \mathbb{R}$ if $|x' - x| < \delta$, then $|f(x') - f(x)| < \epsilon$

The value of the function at x is a good approximation for the value of the function for close by points.

Continuity

This is a local property, we say a function is continuous if it is continuous at every point in its domain.

Continuity

This is a local property, we say a function is continuous if it is continuous at every point in its domain.

Examples

- A line is continuous: $f(x) = ax + b$
  - Consider any $x \in \mathbb{R}$ and some $\epsilon > 0$.
  - Need to find δ such that $|ax' + b - (ax + b)| < \epsilon$.
  - So $\delta = \epsilon/|a|$ works
- Functions can be continuous at some points but not others

$$f(x) = \begin{cases} x \text{ if } x \geq 0 \\ x - 1 \text{ otherwise} \end{cases}$$

is continuous everywhere but 0.

- $x = 0$, $\epsilon > 0$.
- Need to find δ s.t. $|0 - (x' - 1)| < \epsilon$ for all $x' < x$, $|x' - x| < \delta$.
- But for all $x' < x$, $|-x' + 1| > 1$ so this can never hold for any $\epsilon \leq 1$

Continuity

Continuity is preserved under common operations. For instance, if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, then so is $f(x)g(x)$.

▶ Fix x and $\epsilon > 0$.

$$|f(x)g(x) - f(x')g(x')| = |g(x)(f(x) - f(x')) + f(x')(g(x) - g(x)')|$$

▶ Is there a δ s.t. $|g(x)||f(x) - f(x')| + |f(x')||g(x) - g(x)')| < \epsilon$?

▶ We know there's a δ_1 s.t. if $||x - x'|| < \delta_1$

$$|f(x)g(x) - f(x')g(x')| < |g(x)||f(x) - f(x')| + (|f(x)| + 1)|g(x) - g(x)'|$$

.

Continuity

- There's a $\delta < \delta_1$ s.t.

$$|g(x) - g(x')| < \frac{\epsilon}{2(|f(x)| + 1)}$$

  and if $g(x) \neq 0$

$$|f(x) - f(x')| < \frac{\epsilon}{2|g(x)|}.$$

- So if $||x - x'|| < \delta$

$$\begin{aligned}
&|f(x)g(x) - f(x')g(x')| \\
&< |g(x)||f(x) - f(x')| + (|f(x)| + 1)|g(x) - g(x)'| \\
&\leq \epsilon/2 + \epsilon/2
\end{aligned}$$

Continuity

- ▶ Good things: Continuity is preserved under composition, i.e. if f, g are continuous, then so is $f \circ g$.
- ▶ Preserved under things like addition, multiplication etc. $f + g, f - g, f \cdot g$ are continuous if f and g are.
- ▶ A function is continuous iff for every open set E, the set $f^{-1}(E)$ is open

Continuity

There's nothing special about $\mathbb{R}$, we can easily generalize our definition to vectors in $\mathbb{R}^n$.

Definition

A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous at x if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x' \in \mathbb{R}$ if $||x' - x|| < \delta$, then $||f(x') - f(x)|| < \epsilon$

We our now using the norm to measure distance. Everything we've said previously holds here.

Continuity

The fact that we had all these $\delta > 0$'s in both our definition of continuity and convergence should be a hint that there is a connection between the two.

Theorem

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x iff for every convergent sequence $(x_n)_{n=1}^{\infty}$, $x_n \to x$, then $f(x_n) \to f(x)$.

So continuous functions preserve limits.

Continuity

The fact that we had all these $\delta > 0$'s in both our definition of continuity and convergence should be a hint that there is a connection between the two.

Theorem

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x iff for every convergent sequence $(x_n)_{n=1}^\infty$, $x_n \to x$, then $f(x_n) \to f(x)$.

So continuous functions preserve limits. If sequences $x_n \to x$ and

$y_n \to y$, then

- $x_n + y_n \to x + y$
- $x_n y_n \to xy$
- $x_n / y_n \to x/y$ as long as $y, y_n \neq 0$

Some other useful facts about sequences:

- If $x_n \geq y_n$ then $x \geq y$.
- If $x_n > y_n$ then $x \geq y$

Optimization

Now we can go back to our maximization problem:

$$\max_{x \in X} f(x)$$

where $f : \mathbb{R}^m \to \mathbb{R}$. When does a solution to this exist?

Extreme Value Theorem

Theorem (Extreme Value Theorem)

Suppose X is compact and f is continuous. Then $\max_{x \in X} f(x)$ exists.

Proof:

- ▶ Let $c = \sup_{x \in X} f(x)$. This either exists or is $+\infty$. Let's suppose it's finite.
- ▶ There exists a sequence $\{x_n\}$ s.t. $c - f(x_n) < \frac{1}{n}$.
- ▶ By construction $f(x_n) \to c$.
- ▶ X is compact, so $(x_n)_{n=1}^{\infty}$ has a convergent subsequence (x_{n_k}).
- ▶ $f(x_{n_k})$ converges to some $x \in X$, so $f(x_{n_k}) \to f(x) = c$.

(if c is infinite, just replace step 2 with $f(x_n) > n$ to get a contradiction).

Extreme Value Theorem

We have a pretty nice theorem that tells us when a max exists. Just need to check 2 things.

Compactness:

- ▶ Can we find an interval $[a, b]^n$ that contains X. Is X closed?
- ▶ If X is compact, and g is continuous, then so is $g(X)$.
- ▶ $g^{-1}(X)$ is closed if X is closed, g continuous, e.g. $\{x : g(x) \leq 0\}$ is closed

Continuity:

- ▶ Most functions you are used to dealing with, and are commonly used in 1st year economics are continuous.
- ▶ Utility functions, cost functions, etc.
- ▶ But sometimes we run into discontinuous things, e.g. price competition

$$D(p_1, p_2) = \begin{cases} p_1(a - p_1) \text{ if } p_1 < p_2 \\ p_1 \frac{a - p_1}{2} \text{ if } p_1 = p_2 \\ 0 \text{ o.w.} \end{cases}$$

Take-aways

We have a pretty nice theorem that tells us when a max exists

- ▶ Pretty useful, even when we can't compute explictly a max, we know there is one.
- ▶ We can show properties of the max and have them be meaningful
- ▶ "Easy" to check, most things we work with are continuous/compact.
- ▶ Holds in general for problems in $\mathbb{R}^n$.