Notes on Logic and Proofs

Part of this course is learning how to read and write proofs. This will be emphasized, especially in the problem sets. These notes cover some basics of theorems and proofs.

A theorem lays out a set of facts, called *the hypothesis* and a logical implication of these facts called *the conclusion*. These are statements of the form $P \to Q$, in words "If P is true then Q is true". P is the hypothesis, Q is the conclusion. Some theorems:

Theorem 1. *Any number divisible by four is even.*

Theorem 2. *For any numbers $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$*

$$(x+y)^n = \sum_{i=0}^{k} \binom{n}{k} x^k y^{n-k}$$

Theorem 3. *The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.*

We can see in the first theorem the hypothesis is that a number is divisible by 4, and the conclusion is that it is even. In the second, the hypothesis is that $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ and the conclusion is the binomial formula. In the third, the hypothesis is that the triangle is a right triangle and the conclusion is that $a^2 + b^2 = c^2$.

A proof is a string of clear logical implications that lead us from the hypothesis to the conclusion. A problem you'll quickly run into is that it often isn't easy to go directly from the hypothesis to the conclusion. It is often more straightforward to establish different, but logically equivalent, statements. Logically, the statement $P \to Q$ is the same as $\neg Q \to \neg P$ (if Q is false then P is false). This is called the *contrapositive*. This observation is surprisingly powerful for developing proofs. Suppose we want to show that $\sqrt{2}$ is not rational. The natural way to do this is using *proof by contradiction*. But, first we have to figure out how to show that something is not rational.

Theorem 4. *$\sqrt{2}$ is not rational.*

Proof. We are trying to show that $\sqrt{2}$ is not rational. By definition, if $\sqrt{2}$ is rational then there exist $a, b \in \mathbb{Z}$ with no common factors such that $\sqrt{2} = a/b$. So we can take the contrapositive of this definition of rational to get what we want to show, that is if there do not exist a, b such that $\sqrt{2} = a/b$ then $\sqrt{2}$ is not rational. So, the contrapositive gives us a way to prove that $\sqrt{2}$ is not rational, we have to show that no such a, b exist.

It seems hard to show that there are no a, b such that $\sqrt{2} = a/b$ directly. But, now we can do another logical trick. We can assume this is false, and show that that is impossible. Suppose that there exist an $a, b \in \mathbb{Z}$ with no common factors such that $\sqrt{2} = a/b$. Then $2b^2 = a^2$. So a^2 is even, and thus a is even, and can be written as $2k$ for some $k \in \mathbb{Z}$. But then $b^2 = 2k^2$, so b is even (since the square of an odd number is odd). But, then a and b have a common factor, 2, contradicting the assumption that a and b have a common factor. $\square$

Some other contrapositives

- If a number is divisible by 4 then it is even
  - If a number is odd then it is not divisible by 4
- A polygon is a quadrilateral if it has four sides
  - Any polygon that does not have four sides is not a quadrilateral
- A sequence $(x_n)_{n=1}^{\infty}$ is Cauchy if for any $\varepsilon > 0$ there exists an N such that if $n, m \geq N$ then $|x_n - x_m| < \varepsilon$.

– If for some $\varepsilon > 0$ there does not exist an N such that for all $n, m \geq N$ $|x_n - x_m| < \varepsilon$ then the sequence is not Cauchy.

Some other kinds of statements. Defined relative to the statement $P \to Q$.

Type of Statement	**What it means**	**What it means (in English)**	**Some comments**
Negation	P and $\neg Q$	P is true and Q is false	If $P \to Q$ is true then the negation is false. This is how we do proof by contradiction. See above for an example
Inverse	$\neg P \to \neg Q$	If P is false then Q is false.	The inverse of “If a number is divisible by 4 then it is even” is “If a number is not divisible by 4 then it is not even” (false), the inverse of “A polygon is not a quadrilateral if it does not have four sides” (true).
Converse	$Q \to P$	If Q is true then P is true	The negation implies the converse and vice-versa. The converse of the Pythagorean theorem is “If a triangle has sides of length a,b,c that satisfy $a^2 + b^2 = c^2$ then it is a right triangle.” (this is true).